Econ674

Economics of Natural Resources and the Environment

Session 7

Exhaustible Resource

Dynamic Optimization

An Introduction to Exhaustible Resource Pricing

1. The distinction between nonrenewable and renewable resources can become blurred.
2. Renewable resources can be nonrenewable
3. Nonrenewable resources can, in a sense, be renewed through
   - the discovery of new deposits
   - technical advances that make it economically feasible to recover a resource from low-grade materials.
4. We will follow the convention of classifying resources as nonrenewable or renewable depending on the significance of their rates of regeneration in an economic time scale. Thus, e.g., oil is nonrenewable and timber is renewable
5. Are our nonrenewable resources being depleted rapidly/slowly?
6. What is the optimal use of exhaustible resources?

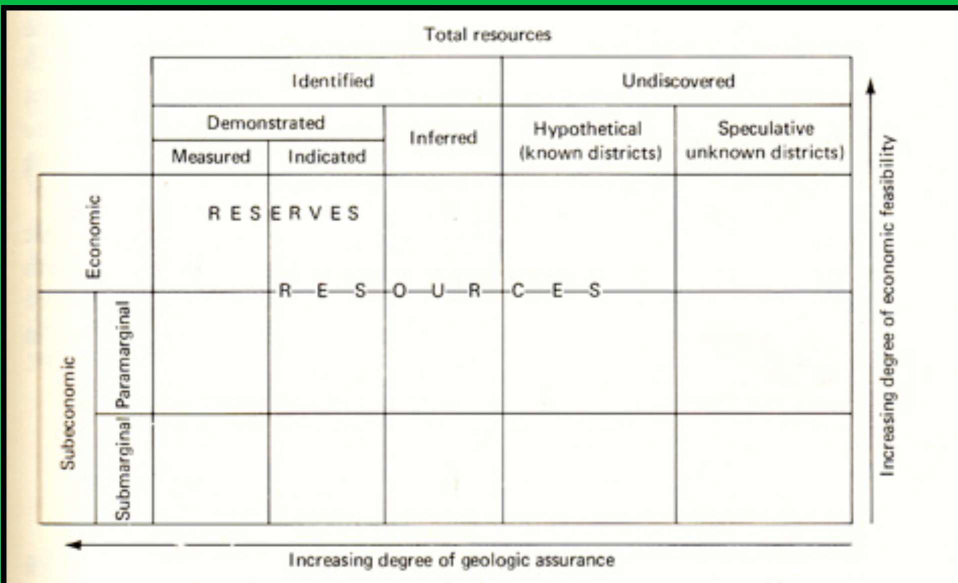
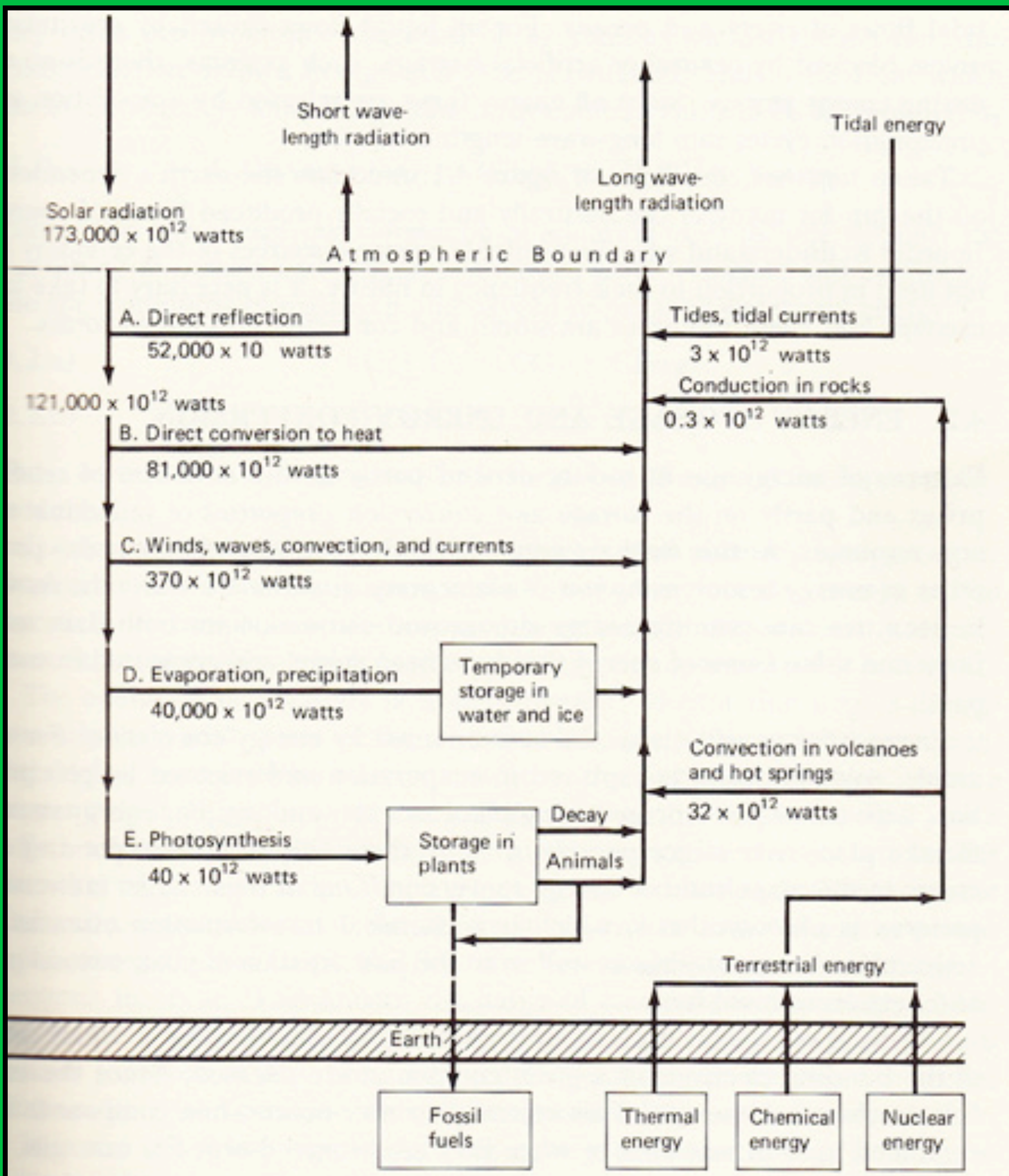


Figure 3.1. Classification of Mineral Resources. Data from U.S. Geological Survey and U.S. Bureau of Mines, *Principles of the Mineral Resource Classification System of the U.S. Bureau of Mines and the U.S. Geological Survey*, Geological Survey Bulletin 1450-A (Washington, D.C.: U.S. Government Printing Office, 1976).

Beyond the use of mineral classification to determine whether a resource is renewable or exhaustible, we also can view the question of time in terms of the earth's energy flows, as shown below.



With this perspective in mind, what steps are involved in analyzing the optimal pricing of exhaustible resources?

A standard approach in such an inquiry is to

1. derive conditions characterizing socially efficient resource use,
2. then establish whether a competitive equilibrium realizes these conditions.
3. Examine the results in terms of whether market pricing achieves an optimal solution, and where market failure arises, devise suitable policy alternatives consistent with an given social welfare function.

Do theorems of welfare economics hold in the context of nonrenewable resources? Imperfections can lead to an inefficient allocation.

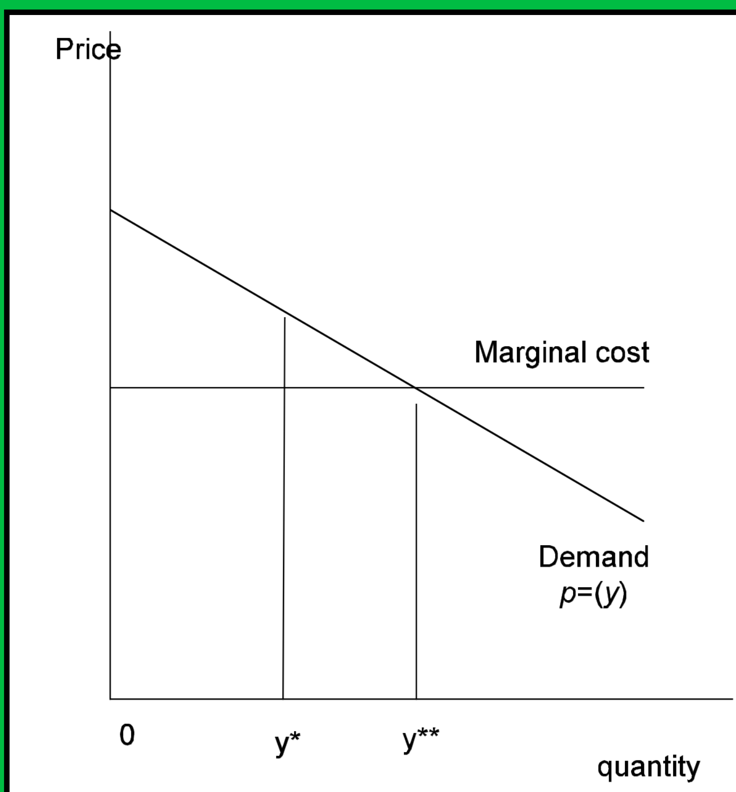
Thus, even if a competitive equilibrium is efficient, we must consider the effects of relevant imperfections and market failures;

- Monopoly
- Environmental disruption in extracting, converting, and use
- Uncertainty surrounding discovery and the long-lasting effects of current decisions about their use.

The theory of optimal exhaustible resources contains two key results:

A) price = MC + opportunity cost. Why should opportunity cost be included? Nonrenewable resources are limited in quantity and are not reproducible. Thus, consumption today has a future opportunity cost that should be taken into consideration.

Accounting for the opportunity cost bears critically on how to allocate exhaustible resources over time, and it recalls the various positions regarding sustainability already discussed. One implication is that when opportunity cost is included, there will be less extraction of a resource today than if it were reproducible.



The relationship between Price and Marginal Cost for an exhaustible resource

Given the demand p = p(y), only y* will be extracted by a planner or resource manager seeking to allocate production efficiently over time, I.e. there will be a positive difference between price and marginal cost.

The difference between price and the marginal extraction cost (MEC) is known variously as the user cost, royalty, rent, net price, or marginal profit. We will use rent, consistent with the taxonomy of factor pricing.

B. The present value (PV) of the rent for an exhaustible resource must be the same in all periods. Equivalently, undiscounted rent must increase at the prevailing rate of interest, or discount.

The net social benefit from extracting a unit of an exhaustible resource is the difference between the market price (or the willingness of consumers to pay) and the marginal extraction cost).

Efficiency Conditions under Pure Depletion for a Competitive Industry

Following Hotelling (1931), under pure depletion, if R(t) represents remaining reserves, and q(t) represents production, then:

$$\dot{R}(t) = -q(t) \qquad \mathbf{EQ}(1)$$

The price of the exhaustible resource, p(t) can be represented as a function whose form is:

$$p(t) = p(0)e^{\delta t} \qquad \mathbf{EQ}(2)$$

Efficiency Conditions under Pure Depletion for a Competitive Industry - 1

Under competitive conditions, the rate of extraction at time t is determined by the following demand function:

$$q(t) = D(p(t)) \qquad \mathbf{EQ}(3)$$

Assuming no extraction costs, initial reserves will be exhausted via the following function:

$$\int_0^T q(t)dt = R \qquad \mathbf{EQ}(4)$$

At t=T, q(T)=0, we then have:

$$q(T) = D\left(p(0)e^{\delta T}\right) = 0 \qquad \mathbf{EQ}(5)$$

Equation 3-5 determines p(0), T, and the entire time-path of extraction. As an example, assume that the demand function D(A) is linear:

$$q(t) = D(p(t)) = a - bp(t) \qquad \mathbf{EQ}(6)$$

thus:

$$q(t) = a - bp(0)e^{\delta t} \qquad \mathbf{EQ}(7)$$

Under the last equation, with t=T and q(T)=0, we have

$$p(0) = ae^{-\delta T} / b$$

Thus:

$$q(t) = a\left(1 - e^{\delta(t-T)}\right) \qquad \mathbf{EQ}(8)$$

Efficiency Conditions under Pure Depletion for a Competitive Industry - 2

Exhaustion of initial reserves implies:

$$\int_0^T a\left(1-e^{\delta(t-T)}\right)dt = R \qquad \mathbf{EQ(9)}$$

Integration yields:

$$aT - \left(1-e^{-\delta T}\right)\frac{1}{\delta}a = R \qquad \mathbf{EQ(10)}$$

Pure Depletion under Monopoly

Let us now examine the behavior of a monopolist owner of an exhaustible resource. As in the competitive initial case, we assume zero marginal extraction costs. The monopolist's objective function can be expressed as:

$$\max \pi = \int_0^{T_m} p\left(q(t)\right)q(t)e^{-\delta t}dt \qquad \mathbf{EQ(11)}$$

Subject to:

$$\dot{R} = -q(t) \quad R(0) = R \qquad \mathbf{EQ(12)}$$

Where p(q(t)) = the inverse of D(p(t)). The monopolist's problem can be formulated as an optimal control problem. The monopolist's current value Hamiltonian may be written as:

$$\overline{H} = p\left(q(t)\right)q(t) - \mu(t)q(t) \qquad \mathbf{EQ(13)}$$

The first-order necessary conditions are:

$$\frac{\partial \overline{H}}{\partial q(t)} = p + q(t)\frac{\partial p(\cdot)}{\partial q(t)} - \mu(t) = 0 \Rightarrow MR(t) = \text{S.P}(\text{t}) \qquad \mathbf{EQ(14)}$$

Pure Depletion under Monopoly - 1

$$\dot{\mu} - \delta\mu(t) = -\frac{\partial \overline{H}}{\partial R(t)} = 0 \qquad \mathbf{EQ(15)}$$

$$\dot{R} = \frac{\partial \overline{H}}{\partial \mu(t)} = -q(t) \qquad \mathbf{EQ(16)}$$

Equation 15 implies that $\dot{\mu}/\mu(t) = \delta$ i.e., the current value shadow price (CV SP) rises at the rate of interest.

However, by equation 14, the CV SP is equated to marginal revenue (MR) at each instant t. Thus the monopolist extracts the resource so that the MR rather than the competitive rent rises at the rate of interest, i.e.:

$$\frac{M\dot{R}(t)}{MR} = \delta \qquad \mathbf{EQ(17)}$$

If we assume a linear demand function, the inverse can be expressed as:

$$p(t) = a/b - q(t)/b \qquad \mathbf{EQ(18)}$$

and the monopolist's MR function thus is defined as:

$$MR = a/b - 2q(t)/b \qquad \mathbf{EQ(19)}$$

In terms of our Hamiltonian, we have: $\overline{H}(T_m) = 0$

i.e., q(Tm)=0 Evaluation equation 14 at t = T_m our inverse demand function implies:

$$\mu(T_m) = a/b - q(T_m)/b \rightarrow \mu(T_m) = a/b \qquad \mathbf{EQ(20)}$$

Pure Depletion under Monopoly - 2

But $\mu(t)=\mu(0)e^{\delta t}$ and $\mu(T_m)=\mu(0)e^{\delta T_m}$ Thus from equation 20, $\mu(0)=ae^{-\delta T_m}/b$ which yields:

$$\mu(t)=ae^{\delta(t-T_m)}/b \qquad \mathbf{EQ(21)}$$

Equating equation 19 with equation 21 and solving for q(t) yields:

$$q(t)=\frac{a}{2}\left(1-e^{\delta(t-T_m)}\right) \qquad \mathbf{EQ(22)}$$

As before, the condition on total reserves gives:

$$\frac{a}{2}T_m - a\left(1-e^{-\delta T_m}\right)/2\delta = R \qquad \mathbf{EQ(23)}$$

Now we compare the exploitation profiles of the competitive and monopolistic industries. If Tc denotes the competitive exhaustion date, we obtain from equation 10 and equation 23:

$T_c-\left(1-e^{-\delta T}\right)/\delta = R/a$ and $T_m-\left(1-e^{-\delta T_m}\right)/\delta = 2R/a$

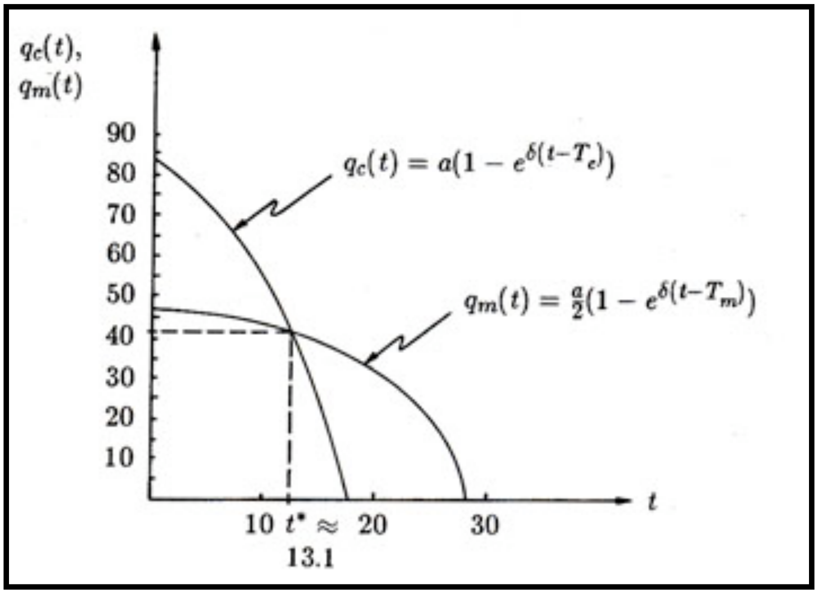
Since these conditions are an increasing function of T, it follows that:

$T_c < T_m$ **Eq (24)**

Equation 22 and 8 show that $q_c(0)>q_m(0)$: Thus, competitive industries exploit the resource at a higher rate initially and exhaust it more rapidly than a monopolist.

An Example of Alternative Extraction Paths under Competition and Monopoly

Let a = 100, b = 10, and R = 1,000 tons.
The resulting values of T are: Tc = 18.41 years and Tm = 29.48 years. The production rates $q_c(t)$ and $q_m(t)$ are shown in the following figure:



the corresponding price paths are:

$P_c(t) = a - bq_c(t)$
$P_m(t) = a - bq_m(t)$.

$P_m(t)$ starts out higher than $p_c(t)$; they intersect at t. 13.1 years and is below $p_c(t)$ until it reaches a choke-off price a/b =10 at T_m = 29.48.

Under these conditions, the monopolist appears as the conservationist's friend (Solow, 1974). Initially, the monopolist restricts production and stretches it out over a longer time horizon.

The monopolist's motivation doesn't derive from concern for future generations, but simply increases the PV by restricting production early on.

The competitive extraction path is socially optimal, and the monopolistic path is dynamically inefficient in the sense that the current generation could more than compensate future generations for an increase in the near term extraction and a reduction in future extraction.

To see how this result is obtained, assume that the social welfare from production q(t) is given by the area under the inverse demand curve given by:

$$U(q(t)) = \int_0^{q(t)} p(z)\,dz \qquad \textbf{EQ(25)}$$

so that the "social" manager would like to :

$$\max = \int_0^T U(q(t))e^{-\delta t}\,dt$$

$$\text{subject to} \quad \dot{R} = -q(t)$$

$$R(0) = R \quad \text{given} \qquad \textbf{EQ(26)}$$

If we construct the current value Hamiltonian, get the first-order conditions and solve, we observe that:

$$U'(q(t)) = p(q(t)) = p(t) = \mu(t)$$

from equation 28, and

$$\dot{\mu} / \mu(t) = \dot{p} / p(t) = \delta$$

From equation 29.

The the welfare maximizing and competitive extraction paths are identical.

How reasonable is the assumption that the competitive price grows at the prevailing rate of interest? For this to be so requires that the owner of the exhaustible resource:

-estimate the date of depletion and choke-off price accurately
- must use the same discount rate

This seems to be an unreasonable assumption for extractive industries in that it requires perfect foresight. It thus ignores:

- price volatility and other forms of uncertainty
- exploration, discovery, and technological change, which suggest the possibility of alternative price paths. Smooth exponentially increasing price paths have not in fact been observed in extractive industries such as mining.

In light of these considerations, let us now consider a few modifications to the basic Hotelling model:

Positive Extraction Costs for a Single Exhaustible Resource
Suppose the cost of mining depends only on the rate of extraction,k ie. C(t) = C(q(t)) Eq. 31
Assume further that p(t) is exogenous and known in advance. The owner of the resource then would:

$$\max = \int_0^T \left[p(t)q(t) - C(q(t))\right]e^{-\delta t}\,dt$$

$$\text{subject to} \quad \dot{R} = -q(t)$$

$$R(0) = R \quad \text{given} \quad R(t) \geq 0 \qquad \mathbf{EQ(32)}$$

The current value Hamiltonian can now be defined as:

$$\overline{H} = p(t)q(t) - C(q(t)) - \mu(t)q(t) \qquad \mathbf{EQ(33)}$$

with first-order necessary conditions that imply:

$$p(t) - C'(q(t)) - \mu(t) = 0 \quad \mathbf{EQ(34)} \;\&\; \dot{\mu} = \delta\mu(t) \quad \mathbf{EQ(35)}$$

Convexity in C(.) implies that the first-order conditions also are sufficient. We thus assume C’(.)>0, and C”(.)>0.

It now can be shown that:

$$\frac{\frac{d}{dt}\left(p(t)-C'(q(t))\right)}{p(t)-C'(q(t))}=\delta \qquad \textbf{EQ(36)}$$

which implies that the price net of marginal cost increases at the prevailing rate of interest.

The corresponding condition for the monopolist's price is

$$\frac{\frac{d}{dt}\left(R'(\cdot)-C'(\cdot)\right)}{R'(\cdot)-C'(\cdot)}=\delta \qquad \textbf{EQ(37)}$$

Equations 36 and 37 tell us about differences in extractive resource owner problems such as how the time path of extraction q(t) is determined.

Geometrically, assume a U-shaped cost curve. The resource owner will never produce at a rate q with)<q<q*. At q*, MC equals AVC.

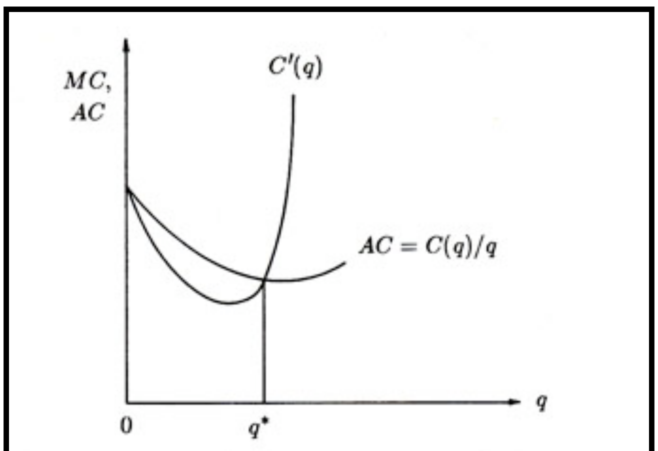


Figure 3.2 Marginal and average cost curves of extraction *q*.

Unless the extraction shuts down temporarily, we have:

$$q^* \le q(t) \quad \text{for } 0 \le t \le T \qquad \textbf{EQ(38)}$$

At time t = T, the extraction operation stops permanently and q drops to zero. The transversality condition for a free-terminal time problem implies:

$$\overline{H} = 0 \quad \text{at } t = T \qquad \textbf{EQ(39)}$$

From equation 33, we obtain:

$$\mu(T) = p(T) - \frac{C(q(T))}{q(T)} \qquad \textbf{EQ(40)}$$

While equation 34 implies that: $\mu(T) = p(T) - C'(q(T))$

Hence:

$$C'(q(T)) = \frac{C(q(T))}{q(T)} \qquad \textbf{EQ(41)}$$

so that: q(T) = q* **EQ (42)**

For a known T horizon, the problem is solved. As equation 40 determines p(T) and by equation 35 we have:

$$\mu(t) = e^{\delta(t-T)}\mu(T) \quad 0 \le t \le T \qquad \textbf{EQ(43)}$$

By equation 34 we have: $C'(q(t)) = p(t) - \mu(t)$
from which q(t) is determined for O < t < T.

T is determined from the condition that R(T) = 0 (Assuming that p(t)>C(q")/q* for all t, which implies that the stock of the resource will be exhausted eventually.

As an example, suppose price p(t) = p (constant), and assume that C(q) = a + bq². Then $q^* = \sqrt{a/b}$ and equations 34-43 imply the conditions:

$$\mu(T) = p - C'(q^*) = p - 2\sqrt{ab} \qquad \mu(t) = e^{\delta(t-T)}\mu(T) \quad 0 \le t \le T$$

$$C'(q(t)) = 2bq(t) = p - \mu(t) \qquad q(t) = \frac{1}{2b}\left[p - e^{\delta(t-T)}\left(p - 2\sqrt{ab}\right)\right]$$

$$\int_0^T q(t)d(t) = \frac{pT}{2b} - \frac{p - 2\sqrt{ab}}{2b}\frac{1 - e^{-\delta T}}{\delta} = R$$

The last equation has a unique solution T>0, so that the problem is solved.

Now consider a discrete-time version of the exhaustible resource owner problem. We can state the problem as:

$$\max_{\{q_t\}} V = \sum_{t=0}^{T-1} \rho^t \left[1 - q_t / R_t\right] q_t$$

$$\text{subject to } R_{t+1} - R_t = -q_t$$

$$R_0 = 1000 \qquad \textbf{EQ}(\textbf{44})$$

For δ = 0.10 and T = 10, v_{10} = 580.2956. Since T assumes integer values, there is no derivative condition for discrete-time problems.

To determine whether it is optimal to lengthen or shorten the extraction horizon one needs to:

- increase or decrease the horizon,
- solve the optimal production schedule (qt),
- calculate the present value of net revenues, Vt, and compare the results.

T	10	20	30	40	50
V_T	580.2956	590.0675	590.3349	590.3423	590.3425

Another look at the Competitive Extractive Industry

Let an extractive industry be comprised of N-price-taking owners. Let:

-Ci (qi(t)) (I=1,2,…N(N)>1) = the cost of extraction for the ith firm

- Ri, (I=1,2,…N (N)>1) = initial reserves, T
- p(t) = the price, which is determined by aggregate production according to:

$$p(t) = p\left(\sum_{i=1}^{N} q_i(t)\right) \qquad \textbf{EQ}(\textbf{45})$$

Each firm attempts to maximize its profits given by:

$$\pi = \int_0^{T_i} \left[p(t) q_i(t) - C_i(q_i(t)) \right] e^{-\delta t} dt \qquad \textbf{EQ}(46)$$

subject to the reserves constraint:

$$\dot{R}_i = -q_i(t) \quad R_i(0) = R_i \quad R_i(T) \geq 0 \qquad \textbf{EQ}(47)$$

From this we need to determine the price path, p = p(t): In the static theory of the firm, we can proceed on the basis of the assumption of competitive markets alone. In a dynamic setting, we must assume that the firm can correctly predict the entire price profile p(t) over time. This assumption is known as rational expectations.

The ith firm's PV (not CV) Hamiltonian can be defined as:

$$H_i=[p(t)q_i(t)-C_i(q_i(t))]e^{-\delta t} - \lambda_i(t)q_i(t) \qquad \textbf{EQ (48)}$$

with necessary conditions that imply:

$$p(t) - C'_i(q_i(t)) = \lambda_i(t) e^{\delta t} \qquad \textbf{EQ}(49)$$

and:

$$\dot{\lambda}_i = -\frac{\partial H_i}{\partial R(t)} = 0 \qquad \textbf{EQ}(50)$$

Thus $\lambda_i(t) = \lambda_i$ a constant. The transversality condition Hi(T)=0 implies:

$$\lambda_i(T) = \left[p(T_i) - C_i(q_i(T_i)) / q_i(T_i) \right] e^{-\delta T_i} \qquad \textbf{EQ}(51)$$

But, evaluating EQ(49) at t = T also implies:

$$\lambda_i = \left[p(T_i) - C'_i(q_i(T_i)) \right] e^{-\delta T_i} \qquad \textbf{EQ}(52)$$

and thus, $q_i(T_i) = q_i^*$ is the level of production where:

$$C'_i(q(T)) = C_i(q_i(T_i)) / q_i(T_i) \qquad \textbf{EQ}(53)$$

that is, where the AVC of the ith resource operation is minimized.

Substituting EQ(53) into EQ(49) implies:

$$C'_i\left(q_i(t)\right) = p(t) - \left[p(T_i) - C'_i\left(q_i(T_i)\right)\right]e^{\delta(t-T_i)} \qquad \mathbf{EQ(54)}$$

At this stage we have 2N+1 unknowns, $T_1,\ldots,T_n$, $q_1(t),\ldots q_N(t)$, and p(t) [$\lambda_1,\ldots,\lambda_N$ can be determined from EQ(51)] and 2n+1 equations consisting of EQ(54) with $q_1(T1)=q_i^*$ plus EQ(45) and

$$\int_0^{T_i} q_i(t)dt = R_i$$

The Social Optimum with N Firms

With No externalities, no stock "common pool" externalities, the perfectly competitive industry would be socially optimal in the sense of maximizing its discounted social welfare (DSW). This can be verified more formally, as defined below:

The social welfare function can be defined as:

$$\max = \int_0^T \left[\left(U\left(\sum_{i=1}^N q_i(t)\right) - \sum_{i=1}^N C_i\left(q_i(t)\right)\right)e^{-\delta t}dt\right]$$

$$\text{subject to } \dot{R}_i = -q_i(t)$$

$$R_i(0) = R_i \quad R_i(t) \geq 0 \qquad \mathbf{EQ(55)}$$

The corresponding Hamiltonian is given by:

$$H = \left(U\left(\sum_{i=1}^N q_i(t)\right) - \sum_{i=1}^N C_i\left(q_i(t)\right)\right)e^{-\delta t} - \sum_{i=1}^N \lambda_i(t)q_i(t) \qquad \mathbf{EQ(56)}$$

Each $\lambda_i(t) = \lambda_i$, a constant (since $\delta H/\delta R_i=0$) and we now have

$$\lambda_i = \left[U'\left(Q(t)\right) - C'_i\left(q_i(t)\right)\right]e^{-\delta t} \qquad \mathbf{EQ(57)}$$

where: $Q(t) = \sum_{t=1}^N q_i(t)$ thus $\partial U/\partial q_i = \partial U/\partial Q \cdot \partial Q/\partial q_i$

and, $U'\left(Q(t)\right) = p\left(Q(t)\right) = p(t)$

If all N firms have identical reserves and costs, $T_i = T$ and $H(T) = 0$

which implies: $\lambda_i = \lambda = \left[p(T) - C\left(q(T)\right)/q(T)\right]e^{-\delta T} \qquad \mathbf{EQ(58)}$

and: $q(T) = q_i^* = q^*$

If the N firms are not identical, it can be shown that:

$$\lambda_i = \left[p(T_i) - C(q_i(T_i))/\, q(T_i)\right]e^{-\delta T_i} \qquad \textbf{EQ(59)}$$

The same system of equations is obtained as in the competitive model under the rational expectations assumption. Both lead to the same solution.

Scarcity Defined in Economic Terms

Economics does not consider scarcity as a physical concept but as a value concept.

The rent for a non-renewable resource is given by the co-state variable:

$$\mu(t) = p(t) - MC(t) \quad \forall t \qquad \textbf{EQ(60)}$$

This reflects the difference between the price and the marginal extraction cost at instant t. In a competitive market, this is the difference between what society would be willing to pay for an additional unit of R(t) and the cost incurred in its extraction. If this difference:

- is positive and large, then the resource is scarce.
- increases over time $(\dot{\mu} > 0)$ i.e., the resource is becoming more scarce
- decreases over time $(\dot{\mu} < 0)$ i.e., the resource is becoming less scarce.

Exploration

Exploration and discovery increase reserves, which may lower extraction costs (e.g., $C(q,R_2)<c(q, R_1)$; $R_1<R_2$). Thus, there are economic incentives to add to known reserves.

Following Pindyck (1978), we state the question of exploration as:

$$\dot{R} = f(w(t), X(t)) - q(t) \qquad \textbf{EQ(61)}$$

and:

$$\dot{X} = f(w(t), X(t)) \qquad \textbf{EQ(62)}$$

where f(.) is a discovery rate as a function of:

-w(t), the exploratory effort, and

-X(t), the level of cumulative discoveries.

Assuming that q(t) is a linear cost function, the total extraction cost per unit of time can be defined as:

$$C_1(R(t))q(t) \quad \textbf{EQ(63)}$$

C1(.) is a unit extraction cost function that is dependent on remaining reserves.

Exploration costs are assumed to be a convex function given as $C_2(w(t))$

Thus, net revenue at instant t, given p(t), is the per unit price of the exhaustible resource:

$$p(t)q(t) - C_1(R(t))q(t) - C_2(w(t)) \qquad \textbf{EQ(64)}$$

Now assume a competitive extractive industry: a large and identical number of firms. The individual firm takes p(t) as exogenous, and under rational expectations, attempts to maximize the following function:

$$\max \int_0^T \left(p(t)q(t) - C_1(R(t))q(t) - C_2(w(t))\right)e^{-\delta t}\,dt \quad \textbf{EQ(65)}$$

$$\textbf{subject to } \dot{R} = f(w(t), X(t)) - q(t)$$

$$\dot{X} = f(w(t), X(t))$$

$$R(0), X(0) \textbf{ given}$$

The problem has:

- two state variables, R, X, and
- two control variables q, w=>, and is more difficult than earlier problems

The current-value Hamiltonian is:

$$\overline{H} = p(t)q(t) - C_1(R)q(t) - C_2(w) + \mu_1[f(\cdot) - q(t)] + \mu_2 f(\cdot) \quad \textbf{EQ(66)}$$

The first-order conditions include:

$$\frac{\partial \overline{H}}{\partial q(t)} = p(t) - C_1(\cdot) - \mu_1(t) = 0 \quad \textbf{EQ(67)}$$

$$\frac{\partial \overline{H}}{\partial w(t)} = -C'_2(\cdot) + (\mu_1(t) + \mu_2(t)) f_w = 0 \quad \textbf{EQ(68)}$$

$$\dot{\mu}_1 - \delta\mu_1 = -\frac{\partial \overline{H}}{\partial R(t)} = C'_1(\cdot)q(t) \quad \textbf{EQ(69)}$$

$$\dot{\mu}_2 - \delta\mu_2 = -\frac{\partial \overline{H}}{\partial X(t)} = -(\mu_1(t) + \mu_2(t)) f_x \quad \textbf{EQ(70)}$$

fx and fw are partials of f() with respect to X(t) and w(t). Taking the time derivative of 67 implies $\dot{\mu}_1(t) = \dot{p}(t) - C'_1(\cdot)\dot{R}$

Substituting this expression into EQ(69) yields:

$$\dot{p}(t) = \delta(p(t) - C_1(\cdot)) + C'_1(\cdot) f(\cdot) \qquad \textbf{EQ(71)}$$

Equation 68 may be solved for μ_2(t) to get:

$$\mu_2(t) = C'_2(\cdot)/f_w - p(t) + C_1(\cdot)$$

Taking the time derive of this expression yields:

$$\dot{\mu}_2 = C''_2(\cdot)\dot{w}/f_w - (f_{w,w}\dot{w} + f_{w,X}\dot{X})C'_2 f_w^{-2} - \dot{p} + C'_1 \dot{R} \qquad \textbf{EQ(72)}$$

Substituting the expression for μ_2(t) into 70 $[\dot{\mu}_2 = \delta\mu_2 - (\mu_1(t) + \mu_2(t))f_x]$ and noting from 68 that $(\mu_1(t) + \mu_2(t)) = C'_2(\cdot)/f_w$

We obtain:

$$\dot{\mu}_2 = \delta(C'_2(\cdot)/f_w - p(t) + C_1(\cdot)) - C'_2(\cdot) f_x / f_w \qquad \textbf{EQ(73)}$$

Equating 72 and 73 and solving for the rate of change in w yields:

$$\dot{w} = \left\{ f_{w,X}\dot{X}\frac{C'_2}{f_w^2} + \dot{p} - C'_1\dot{R} + \delta\frac{C'_2(\cdot)}{f_w} - \delta p(t) + \delta C_1(\cdot) - C'_2(\cdot)\frac{f_x}{f_w} \right\} \left(\frac{f_w^2}{f_w C''_2(\cdot) - f_{w,w}C'_2} \right) \qquad \mathbf{EQ}(\mathbf{7}4)$$

By substituting 3.70 for the rate of change in p and simplifying:

$$\dot{w} = \left[\frac{\left[\left(f_{w,X}/f_w \right) f(\cdot) + \delta - f_x \right] C'_2 + C'_1(\cdot)q(t)f_w}{\left(C''_2(\cdot) - f_{w,w}C'_2/f_w \right)} \right] \qquad \mathbf{EQ}(\mathbf{7}5)$$

Equations 61, 62, 71, and 74 are a four-equation dynamical system for the five unknown functions R(t), X(t), w(t), p(t), and q(t). The latter function may be eliminated by the demand equation:

q(t) = D(p(t))

The terminal conditions for the rates of change in p and w depend on C'_2/f_w as t => T.

This expression defines the ratio of the marginal cost of exploration to the marginal product of exploratory effort, and is referred to as the "marginal discovery cost".

If C2'(O)/fw(0,X)=0, then w(T+ q(T) = 0 simultaneously at t = T. It will also be the case that : $\mu_2(T) = 0$ and $\mu_1(T) = p(T) - C_1(R(T)) = 0$

This means that no additional profit can be obtained from further extraction.

If $C'_2(0)/f_w(0,X) = \phi > 0$ then exploratory effort will become 0 before extraction, I.e., there will exist an interval T1<t<T where w(t) = 0, but q(t) >0.

At t = T, $\mu_2(T1) = 0$ and with w(t) = 0, $\dot{\mu}_2 = 0$ Then for all t in T,

$$p(t) - C_1(\cdot) = \mu_1 = \phi$$

This implies $\dot{\mu}_1 = 0$ and that $-C'_1(\cdot)q(t)/\delta \to \phi$ at t approaches T. thus for all t in T, both p(t) - C_1() and c'_1(.)q(t) remain constant, implying p(t), C_1(), and c_1'() rise as q(t) falls.

The last unit of reserves should be discovered when its MC of discovery equals the sum of the net revenue obtained upon extraction, and sale, and the value of cost savings after discovery, but before extraction. This sum is the PV of net revenue of the marginal discovery.

The system $\dot{R}, \dot{X}, \dot{p}, \text{ and } \dot{w}$ leads to several dynamic possibilities.

In graphical terms, we may portray the dynamic effects in terms of the following graphs shown below:

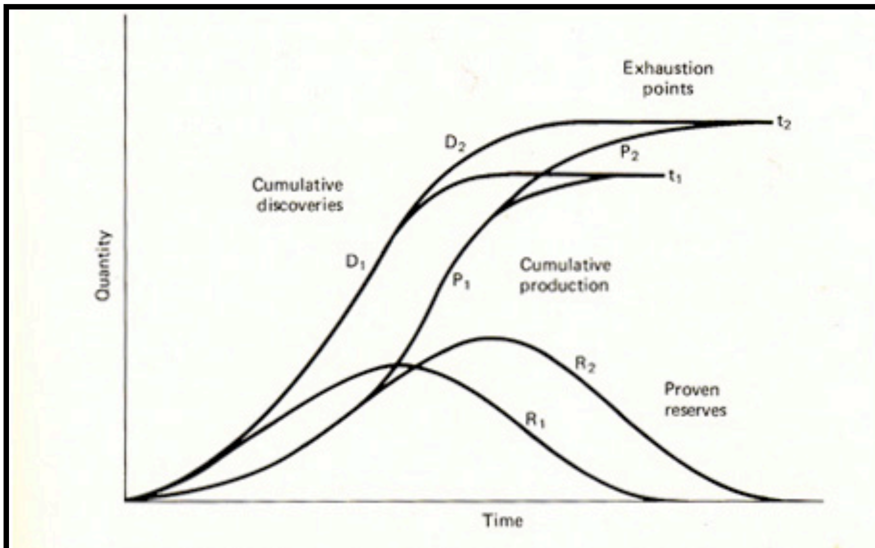


Figure 3.3.*a*. The Life Cycle of an Exhaustible Resource with Increased Supply

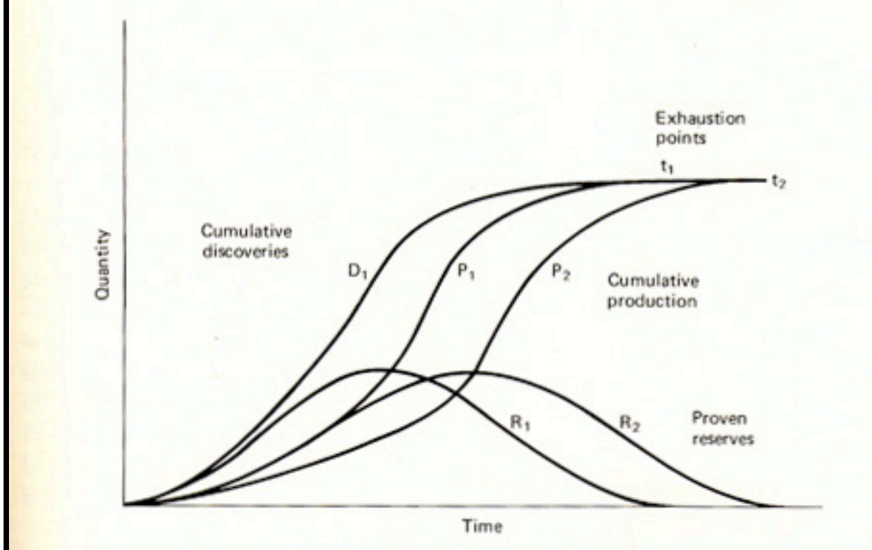


Figure 3.3.*b*. The Life Cycle of an Exhaustible Resource with Demand Restraint

Using Discrete-Time Models to Solve for the Optimal Rate of Extraction of an Exhaustible Resource

Discrete Time Models of Exhaustible Resources with Variable Time Horizons

P. LeBel

The efficient use of an exhaustible resource is based on a multi-period framework in which the user cost, or rent, increases at the prevailing rate of interest. While there are many formulations that make use of this fundamental principle, we will illustrate here the use of a discrete-time period model for a competitive resource. For a given quantity of the exhaustible resource, the solution to an n-period model can be derived in terms of a constrained optimization model, the matrix framework for which is outlined in the following applications. As with any investment, the higher is the discount rate, the higher will be the rate of extraction in the present relative to the future. The conservation of a resource is thus consistent with a low rate of discount.

Simulation Tableau

	Simulation Values	Base Case
Quantity of Exhaustible Resource	150.00	150.00
Base Period Demand Intercept	10.00	10.00
Demand Curve Slope	-0.0500	-0.0500
Rate of Increase in Demand	0.00%	0.00%
Rate of Discount	5.00%	5.00%
Extraction Cost Rate	0.0000	0.0000
NPV-Two Period Model	$914.63	$914.63
NPV-Three Period Model	$1,070.58	$1,070.58
NPV-Four Period Model	$1,131.06	$1,131.06

A. Two-Period Model:

Extraction Cost Assumption: $0.00 Discount Rate: 5.00%

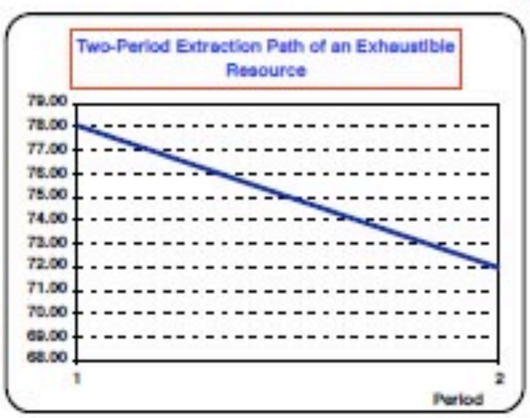
Demand:				
	Po =	10.00	-0.0500	Q0
	P1 =	10.00	-0.0500	Q1
	Q0 + Q1 =	150.00		

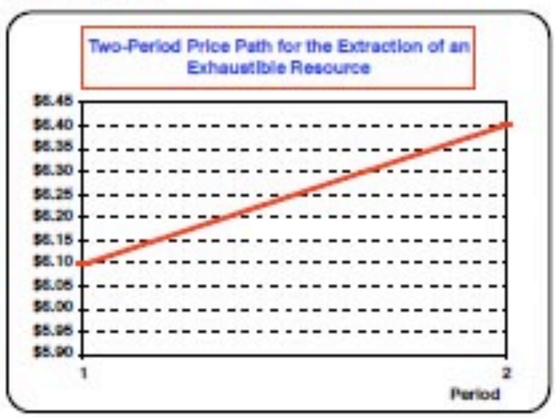
for: AB = C, then: (A^-1)C = B

Solution Matrix:

	A: Q0	Q1	Lambda	C:	
Q0	0.0500	0.0000	1.0000	10.0000	=Q0
Q1	0.0000	0.0476	1.0000	9.5238	=Q1
Lambda	1.0000	1.0000	0.0000	150.00	

	A^-1 Q0	Q1	Lambda	C:	B:		UCi:	
Q0	10.2439	-10.2439	0.4878	10.0000	78.05	=Q0	$6.10	
Q1	-10.2439	10.2439	0.5122	9.5238	71.95	=Q1	$6.40	5.00% Rate of Increase
Lambda	0.4878	0.5122	-0.0244	150.00	6.10	=Lambda, or UC(o)		
				Q0 + Q1 =	150.00			





Using Discrete-Time Models to Solve for the Optimal Rate of Extraction of an Exhaustible Resource - 1

B. Three-Period Model:

Other things equal, an increase in the number of time periods will reduce the consumption in any given period. In addition, the base period price will be correspondingly higher than where a reduced number of time periods is used.

Extraction Cost Assumption: $0.00 **Discount Rate:** 5.00%

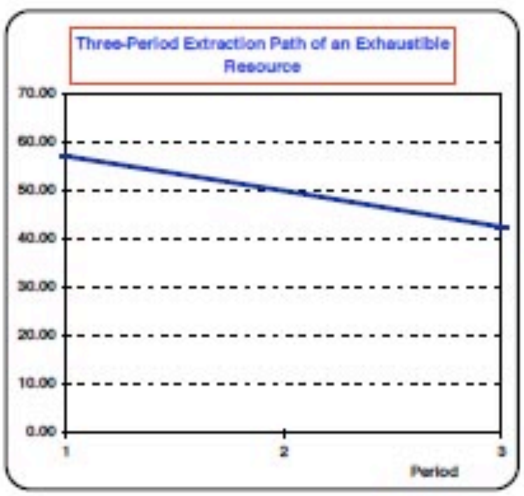
Demand:			
	Po =	10.00	-0.0500 Q0
	P1 =	10.00	-0.0500 Q1
	P2 =	10.00	-0.0500 Q2
	Q0+Q1+Q2 =	150.00	

for: AB = C, then: (A^-1)C = B

Solution Matrix:

	A:				C:	
	Q0	Q1	Q2	Lambda		
Q0	0.0500	0.0000	0.0000	1.0000	10.00	=Q0
Q1	0.0000	0.0476	0.0000	1.0000	9.5238	=Q1
Q2	0.0000	0.0000	0.0454	1.0000	9.0703	=Q2
Lambda	1.0000	1.0000	1.0000	0.0000	150.00	=Lambda

	A^-1				C:	B:		UCi:	
	Q0	Q1	Q2	Lambda					
Q0	13.6558	-6.6614	-6.9944	0.3172	10.0000	57.26	=Q0	$7.14	
Q1	-6.6614	14.0056	-7.3442	0.3331	9.5238	50.12	=Q1	$7.49	5.00% rate of increase
Q2	-6.9944	-7.3442	14.3386	0.3497	9.0703	42.62	=Q2	$7.87	5.00% rate of increase
Lambda	0.3172	0.3331	0.3497	-0.0159	150.0000	7.14	=Lambda, or UC(o)		
					Q0+Q1+Q2 =	150.00			



Using Discrete-Time Models to Solve for the Optimal Rate of Extraction of an Exhaustible Resource - 2

C. **Four-Period Model:**

Demand:			
P0 =	10.00	-0.0500	Q0
P1 =	10.00	-0.0500	Q1
P2 =	10.00	-0.0500	Q2
P3 =	10.00	-0.0500	Q3

Discount Rate:	5.00%
Demand Growth Rate:	0.00%
Extraction Cost Assumption:	$0.00
Q0+Q1+Q2+Q3=	150.00

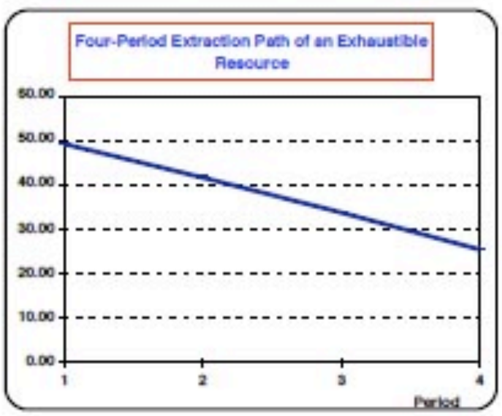
for: AB = C, then: (A^-1)C = B

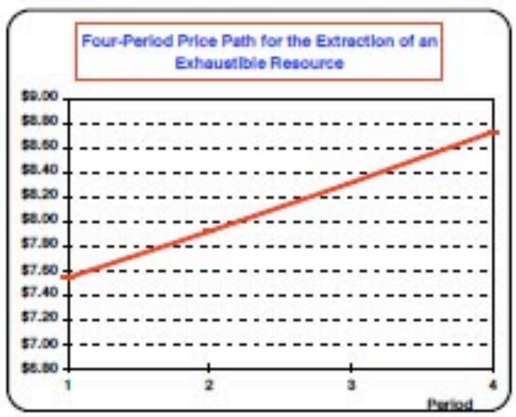
Solution Matrix:

	A:					C:	
	Q0	Q1	Q2	Q3	Lambda		
Q0	0.0500	0.0000	0.0000	0.0000	1	10.00	=Q0
Q1	0.0000	0.0476	0.0000	0.0000	1	9.52	=Q1
Q2	0.0000	0.0000	0.0454	0.0000	1	9.07	=Q2
Q3	0.0000	0.0000	0.0000	0.0432	1	8.64	=Q3
Lambda	1.0000	1.0000	1.0000	1.0000	0	150.00	=Lambda

	A^-1					C:	B:		UCI =	
	Q0	Q1	Q2	Q3	Lambda					
Q0	15.3598	-4.8722	-5.1159	-5.3717	0.2320	10.00	**49.19**	=Q0	**$7.54**	
Q1	-4.8722	15.8841	-5.3717	-5.6402	0.2436	9.52	**41.65**	=Q1	**$7.92**	5.00% rate of increase
Q2	-5.1159	-5.3717	16.4098	-5.9222	0.2558	9.07	**33.73**	=Q2	**$8.31**	5.00% rate of increase
Q3	-5.3717	-5.6402	-5.9222	16.9341	0.2686	8.64	**25.42**	=Q3	**$8.73**	5.00% rate of increase
Lambda	0.2320	0.2436	0.2558	0.2686	-0.0116	150.00	7.54	=Lambda, or UC(o)		

Q0+Q1+Q2+Q3= 150.00





Implications of multi-period exhaustible resource models:

1. The longer is the time horizon for the consumption of an exhaustible resource, the higher will be the initial price and initial user cost. Although technological change can produce an augmentation in the level of proven reserves, there is no guarantee that this will solve the problem of an efficient allocation of resources over time, which is what gives rise to conservation arguments purely on the grounds of preservation for future generations. This is the argument put forth in classical economic models in which the steady-state was viewed as an uncertain time horizon that called for increased conservation. In response, neoclassical models are based on the principle of resource substitution, as in the introduction of backstop technologies.
2. Beyond these considerations is the fact that if one knew what the terminal time period were, one also would know something about the end of the economic world as we know it. The reality is that we do not, which is why so much emphasis in neoclassical models is placed on resource substitution. Indeed, this was one basis of critiques of William Stanley Jevons' 1865 study, *The Coal Question*, and the 1970 Club of Rome Report, *The Limits to Growth*.
3. One additional question is that none of these models takes up directly the negative externality question of what is the optimal consumption path for an exhaustible resource when environmental emissions are present. Other models do take up this question, which leads to discussions of sustainability in which climate change is explicitly taken into consideration.

Using Discrete-Time Models to Solve for the Optimal Rate of Extraction of an Exhaustible Resource - 3

Optimal Present Values of User Costs of a Two-Period Exhaustible Resource

The optimal allocation of an exhaustible resource is one that results in the user cost rising at the prevailing rate of discount. It also is the one that maximizes the present value of the user costs across the investment time horizon. Using the two-period example, we compare here the optimal solution with alternative proportional allocations of the fixed stock of the resource to illustrate the effects on the present value (PVUCi) of the resource over the two time periods.

Extraction Cost Assumption: $0.00 **Discount Rate:** 5.00%

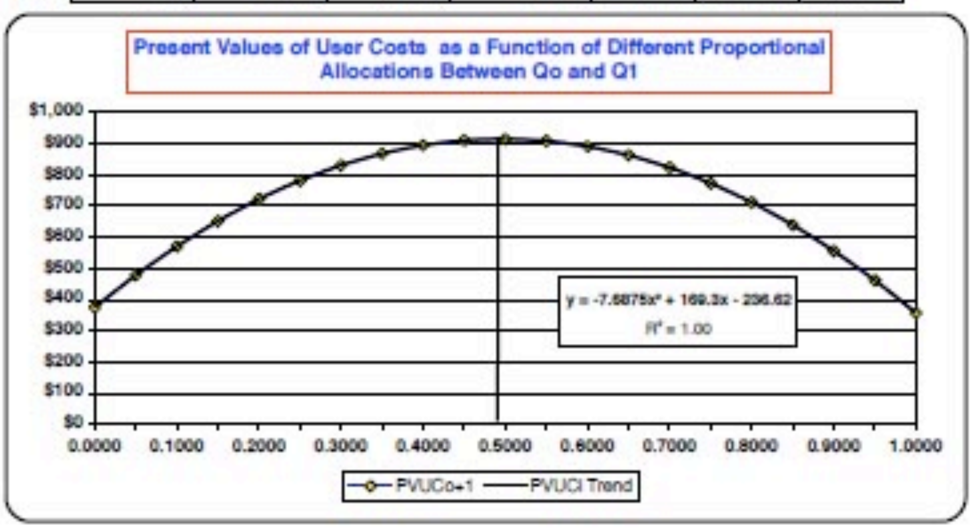
Demand:
$P_0 =$ 10.00 -0.0500 Q0
$P_1 =$ 10.00 -0.0500 Q1

$Q_0 + Q_1 =$ **150.00**

Q1 share	Q_0	Q_1	UC_0	UC_1	$PVUC_0$	$PVUC_1$	$PVUC_{0+1}$
0.000	150.00	0.00	$2.50	$10.00	$375.00	$0.00	$375.00
0.050	142.50	7.50	$2.88	$9.63	$409.69	$68.75	$478.44
0.100	135.00	15.00	$3.25	$9.25	$438.75	$132.14	$570.89
0.150	127.50	22.50	$3.63	$8.88	$462.19	$190.18	$652.37
0.200	120.00	30.00	$4.00	$8.50	$480.00	$242.86	$722.86
0.250	112.50	37.50	$4.38	$8.13	$492.19	$290.18	$782.37
0.300	105.00	45.00	$4.75	$7.75	$498.75	$332.14	$830.89
0.350	97.50	52.50	$5.13	$7.38	$499.69	$368.75	$868.44
0.400	90.00	60.00	$5.50	$7.00	$495.00	$400.00	$895.00
0.450	82.50	67.50	$5.88	$6.63	$484.69	$425.89	$910.58
0.500	75.00	75.00	$6.25	$6.25	$468.75	$446.43	$915.18
0.550	67.50	82.50	$6.63	$5.88	$447.19	$461.61	$908.79
0.600	60.00	90.00	$7.00	$5.50	$420.00	$471.43	$891.43
0.650	52.50	97.50	$7.38	$5.13	$387.19	$475.89	$863.08
0.700	45.00	105.00	$7.75	$4.75	$348.75	$475.00	$823.75
0.750	37.50	112.50	$8.13	$4.38	$304.69	$468.75	$773.44
0.800	30.00	120.00	$8.50	$4.00	$255.00	$457.14	$712.14
0.850	22.50	127.50	$8.88	$3.63	$199.69	$440.18	$639.87
0.900	15.00	135.00	$9.25	$3.25	$138.75	$417.86	$556.61
0.950	7.50	142.50	$9.63	$2.88	$72.19	$390.18	$462.37
1.000	0.00	150.00	$10.00	$2.50	$0.00	$357.14	$357.14

Original Optimal Solution

	Q_i	UC_i	$PVUC_i$
Q_0	**78.05**	**$6.10**	**$475.91**
Q_1	**71.95**	**$6.40**	**$438.73**
	150.00		**$914.63**
Q1%	**47.97%**	**Optimal Proportion**	



Using Discrete-Time Models to Solve for the Optimal Rate of Extraction of an Exhaustible Resource - 4

Variations on the Competitive Model Exhaustible Resources

 In a two-period competitive model, the present value of the user cost, λ, must be equalized for each time period. P. LeBel

Consider the two-period model below. For each time period, an inverse demand function of the form Pi = a - bQi is given. If we subtract from each period's inverse demand function, the corresponding marginal extraction cost (MEC), we obtain the user cost for each time period. The problem is to allocate production across the two time periods such that the user cost increases at the present rate of interest (or discount rate), subject to the overall stock of the exhaustible resource.

		(a	- b	Qi) -	MEC (Qi)		= UC_i
1.	P_0 =	60.00	-0.0500	Q_0	-0.0000	Q_0	= λ
2.	P_1 =	60.00	-0.0500	Q_1	-0.0000	Q_1	= λ

3. $Q_0 + Q_1$ = 150.00 = the stock of an exhaustible resource.

4. Discount Rate: 10.00% We first state equation 1 in terms of the given user cost:

5. UC_0 = 60.00 -0.0500 Q_0 = λ In the second time period, the user cost function must be discounted by the prevailing rate of discount of: 10.00%, which is used to derive a present worth coefficient (PWC) that can then be multiplied by each term on the left-hand side in equation 2. Given the rate of discount, our PWC equals: 0.9091 which when applied to equation 2 yields:

6. UC_1 = 54.55 -0.0455 Q_1 = λ Next, since equations 5 and 6 equal lambda, λ, set them equal:

7. 60.00 -0.0500 Q_0 = 54.55 -0.0455 Q_1 Rearranging and simplifying we obtain:

8. 5.45 +0.0455 Q_1 = +0.0500 Q_0 Simplifying, we obtain:

9. 109.09 +0.9091 Q_1 = 1.0000 Q_0 By substitution of the left-hand equation into equation 3, we obtain:

10. 109.09 +0.9091 Q_1 + 1.0000 Q_1 = 150.00 Substituting Q's into the UC's gives:

11. 1.9091 Q_1 = 40.91 | 12. Q_1 = 21.43 | UC_1 = $58.93 | 10.00% = rate of increase in user cost

By substituting into equation 3, we obtain: 13. Q_0 = 128.57 | UC_0 = $53.57, which satisfies the efficiency criterion.

Consider now the effect of a different discount rate of: 5.00% which gives a PWC of 0.9524 which gives:

14. UC_1 = 57.14 -0.0476 Q_1 = λ which we then use to repeat the steps starting in equation 7:

15. 60.00 -0.0500 Q_0 = 57.14 -0.0476 Q_1 Simplifying, we obtain:

16. 2.86 +0.0476 Q_1 = +0.0500 Q_0 Simplifying we obtain:

17. 57.14 +0.9524 Q_1 = +1.0000 Q_0 which through subsitution yields:

18. 57.14 +0.9524 Q_1 + (1.00)Q_1 = 150.00 Substituting Q's into the UC's gives:

19. 1.9524 Q_1 = 92.86 | 20. Q_1 = 47.56 | UC_1 = $57.62 | 5.00% = rate of increase in user cost

which when substituted into equation 3: 21. Q_0 = 102.44 | UC_0 = $54.88, which satisfies the efficiency criterion.

Consider now the effect of an increase in demand by: 5.00% Equation 2 now becomes:

22. P_1 = 63.00 -0.0500 Q_1 -0.0000 Q_1 = λ Using the **original** discount rate,

23. UC_1 = 57.27 -0.0455 Q_1 = λ which when set equal to equation 5 yields:

24. 60.00 -0.0500 Q_0 = 57.27 -0.0455 Q_1 Simplifying, we obtain:

25. 2.73 +0.0455 Q_1 = +0.0500 Q_0

26. 54.55 +0.9091 Q_1 = 1.0000 Q_0

27. 54.55 +0.9091 Q_1 + 1.0000 Q_1 = 150.00

28. 1.9091 Q_1 = 95.45 | 29. Q_1 = 50.00 | UC_1 = $60.50 | 10.00% = rate of increase in user cost

30. Q_0 = 100.00 | UC_0 = $55.00

Consider now technical change to increase reserves at: 3.00% While P_0 remains the same, technological change produces the following:

31. UC_1 = (60.00 -0.0500 Q_1 -0.0000 Q_1) x 1.03 = 61.80 -0.0515 Q_1

32. UC_1 = 61.80 -0.0515 Q_1 ÷ 1.1000 = 56.18 -0.0468 Q_1

33. $Q_0 + Q_1$ = 150.00 + 0.03 Q_1

34. 60.00 -0.0500 Q_0 = 56.18 -0.0468 Q_1, which by rarranging and simplifying we obtain:

35. 3.82 +0.0468 Q_1 = 0.0500 Q_0 Simplifying, we obtain:

36. 76.36 +0.9364 Q_1 = 1.0000 Q_0 Substituting into equation 32, we obtain:

37. 76.36 +0.9364 Q_1 + 1.0000 Q_1 = 150.00 ###### Q_1 Simplifying,

38. 1.91 Q_1 = 73.64 Thus:

39. Q_1 = 38.63 | UC_1 = $59.81 | 10.00% = rate of increase in user cost

Since Q1 has a 3.00% change over the base, we can obtain Q_0 by adjusting equation 44 to obtain Q_0:

Q_0 = 150.00 -(0.9700)x 38.63 i.e., 40. Q_0 = 112.53 | UC_0 = $54.37

$Q_0 + Q_1$ = 151.16, the new augmented quantity of reserves.

Finally, consider a backstop technology priced at: $6.25, and which becomes available in period 1. Our problem now becomes:

41. P_0 = UC_0 = 60.00 -0.0500 Q_0 = ($6.25) ÷ 1.10 = $5.68

By substitution, we solve first for Q_0 to obtain: 42. Q_0 = 1,086.36 | UC_0 = $5.68 And by substitution into Equation 3,

43. Q_1 = -936.36 | UC_1 = $6.25 | 10.00% = rate of increase in user cost

As long as the backstop technology price is below what the exhaustible model would have generated, production will be shifted from the future to the present to capture the economic value of the exhaustible resource.