TRANSCENDENTAL LOGARITHMIC PRODUCTION FRONTIERS

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I Introduction

ADDITIVE and homogeneous production possibility frontiers have played an important role in formulating statistical tests of the theory of production. In section II we characterize the class of production possibility frontiers that are homogeneous and additive. This class coincides with the class of frontiers with constant elasticities of substitution. Constancy of the elasticity of substitution has proved to be a fruitful point of departure for the analysis of production with one output and two factors of production, as in the pioneering study of capital-labor substitution by Arrow, Chenery, Minhas, and Solow.[1] For more than one product or more than two factors of production, constancy of elasticities of substitution and transformation is highly restrictive, as Uzawa and McFadden have demonstrated.[2]

Our first objective is to develop tests of the theory of production that do not employ additivity and homogeneity as part of the maintained hypothesis. For this purpose we introduce new representations of the production possibility frontier in section III. Our approach is to represent the production frontier by functions that are quadratic in the logarithms of the quantities of inputs and outputs. These functions provide a local second-order approximation to any production frontier. The resulting frontiers permit a greater variety of substitution and transformation patterns than frontiers based on constant elasticities of substitution and transformation.[3]

A complete model of production includes the production possibility frontier and necessary conditions for producer equilibrium. Under constant returns to scale this model implies the existence of a price possibility frontier, defining the set of prices consistent with zero profits.[4] Necessary conditions for producer equilibrium, giving relative prices as a function of relative product and factor intensities, imply the existence of conditions determining relative product and factor intensities as a function of relative prices. The price possibility frontier and the conditions determining product and factor intensities are dual to the production possibility frontier and the necessary conditions for producer equilibrium.[5]

Our second objective is to exploit the duality between prices and quantities in the theory of production. Our approach is to represent the price possibility frontier by functions that are quadratic in the logarithms of prices, paralleling our treatment of the production possibility frontier. These functions provide a local second-order approximation to any price frontier. The duality between direct and indirect utility functions employed in Houthakker's pathbreaking studies of consumer demand is analogous to the duality between production and price frontiers employed in our study of production.[6]

We refer to our representation of the production possibility frontier as the transcen-

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[1] See Arrow, Chenery, Minhas, and Solow (1961). A review of the literature on capital-labor substitution is given by Jorgenson (1972).

[2] See Uzawa (1962) and McFadden (1963).

[3] An alternative representation that also permits a variety of substitution and transformation patterns is the "generalized Leontief" production possibility frontier proposed by Diewert (1971, 1973).

[4] The price possibility frontier was introduced by Samuelson (1953). For a single output and many inputs, Samuelson referred to the price possibility frontier as the factor-price frontier.

[5] Duality between the production and price possibility frontiers is discussed by Samuelson (1953), pp. 15–20, Bruno (1969), and Burmeister and Kuga (1970).

[6] See Houthakker (1960).

dental logarithmic production possibility frontier or, more simply, the *translog production frontier.* The production possibility frontier is a transcendental function of the logarithms of its arguments, the quantities of net outputs. Similarly, we refer to our representation of the price possibility frontier as the transcendental logarithmic price possibility frontier or, more simply, the *translog price frontier.*[7] For many of the production and price frontiers employed in econometric studies of production the translog frontiers provide accurate global approximations.[8] The accuracy of the approximation must be determined separately for each application.

We present statistical tests of the theory of production in section IV. These tests can be divided into two groups. First, we test restrictions on the parameters of the translog production frontier implied by the theory of production. We test these restrictions without imposing the assumptions of additivity and homogeneity. We test precisely analogous restrictions on the parameters of the translog price frontier. Second, we test restrictions on the translog production frontier corresponding to restrictions on the form of the frontier. In particular, we test restrictions on the form of technical change and restrictions implied by the assumption of additivity. Again, we test precisely analogous restrictions on the translog price frontier.

We present empirical tests of the theory of production, based on time series data for the United States private domestic economy for 1929–1969 in section V. The data include prices and quantities of investment and consumption goods output and labor and capital services input and an index of total factor productivity. For these data we present direct tests of the theory of production, based on the translog production frontier, and indirect tests of the theory, based on the translog price frontier. For both direct and indirect tests our empirical results are consistent with a very extensive set of restrictions implied by the theory of production. Proceeding conditionally on the validity of the theory of production, our empirical results are inconsistent with restrictions on the form of the production frontier implied by the assumption of additivity.

II Additivity and Homogeneity

1) Introduction

Our purpose in this section is to derive the implications of additivity and homogeneity for the representation of the production possibility frontier. The class of additive and homogeneous production possibility frontiers coincides with the class of frontiers with constant elasticities of substitution and transformation. We first represent the production possibility frontier in the form:

$$F(y_1, y_2 \ldots y_n) = 0, \tag{1}$$

where y_i $(i = 1, 2 \ldots n)$ represents the net output of the i^{th} commodity.

The necessary conditions for producer equilibrium take the form of equalities between price ratios and marginal rates of transformation between the corresponding pair of commodities:

$$\frac{q_i}{q_j} = \frac{\dfrac{\partial F}{\partial y_i}}{\dfrac{\partial F}{\partial y_j}} \qquad (i \neq j;\ i, j = 1, 2 \ldots n), \tag{2}$$

where q_i $(i = 1, 2 \ldots n)$ represents the price of the i^{th} commodity.

2) Commodity-wise Additivity

The production possibility frontier is characterized by constant returns to scale if and only if:

[7] The translog production and price possibility frontiers were introduced by Christensen, Jorgenson, and Lau (1971); independently, Griliches and Ringstad (1971) and Sargan (1971) proposed a special case of the translog production frontier, the translog production function. The translog production function of Griliches and Ringstad and Sargan has only a single output.

[8] Kmenta (1967) employed a special case of the translog production function to approximate the constant elasticity of substitution (CES) production function of Arrow, Chenery, Minhas, and Solow (1961). His approximation is in the space of parameters rather than the space of variables. An approximation to the CES production function that is quadratic in the logarithms of the variables is identical to Kmenta's approximation. See below, section IV, 5.

Kmenta's approximation to the CES production function, considered as a production function in its own right, is a homogeneous translog production function. Similarly, the log-quadratic production function proposed by Chu, Aigner, and Frankel (1970) can be regarded as a commodity-wise additive but not homogeneous translog production function.

$$F(\lambda y_1, \lambda y_2 \ldots \lambda y_n) = F(y_1, y_2 \ldots y_n) = 0, \quad (3)$$

for any $\lambda > 0$. We refer to production possibility frontiers satisfying this condition as *homogeneous*. The production possibility frontier is *commodity-wise additive* if and only if the frontier can be represented in the form:

$$F(y_1, y_2 \ldots y_n) = F^1(y_1) + F^2(y_2) + \ldots + F^n(y_n) = 0, \quad (4)$$

where the functions $[F^i]$ are strictly monotone and depend on only a single variable.[9]

The production possibility frontier is homogeneous and commodity-wise additive if and only if the frontier can be represented in the form:

$$F^1(\lambda y_1) + F^2(\lambda y_2) + \ldots + F^n(\lambda y_n) = F^1(y_1) + F^2(y_2) + \ldots + F^n(y_n) = 0. \quad (5)$$

But the frontier satisfies this condition if and only if:

(1) The functions $[F^i]$ are homogeneous of the same degree, or:

(2) The functions $[F^i]$ are logarithmic.[10]

Any homogeneous function of one variable can be represented in the form:

$$F^i(y_i) = (\text{sgn } y_i)\, a_i |y_i|^\rho, \quad (6)$$

where ρ is the degree of homogeneity and $a_i > 0$. If the functions $[F^i]$ are homogeneous of the same degree, the production possibility frontier can be represented in the form:

$$F(y_1, y_2 \ldots y_n) = (\text{sgn } y_1)\, a_1|y_1|^\rho + (\text{sgn } y_2)\, a_2|y_2|^\rho + \ldots + (\text{sgn } y_n)\, a_n|y_n|^\rho = 0, \quad (7)$$

where:

$$(\text{sgn } y_1)\, a_1 + (\text{sgn } y_2)\, a_2 + \ldots + (\text{sgn } y_n)\, a_n = 0.$$

For this frontier to have the proper curvature there can be only one output ($\rho < 1$) or only one input ($\rho > 1$), unless all net outputs are perfect substitutes ($\rho = 1$). For only one output the frontier is characterized by constant elasticities of substitution between inputs; for one input the frontier has constant elasticities of transformation between outputs.[11]

Alternatively, if the functions $[F^i]$ are logarithmic,

$$F^i(y_i) = (\text{sgn } y_i)\, a_i \ln|y_i|, \qquad (i = 1, 2 \ldots n), \quad (8)$$

where $a_i > 0$ $(i = 2 \ldots n)$, as before. The production possibility frontier can be represented in the form:

$$F(y_1, y_2 \ldots y_n) = (\text{sgn } y_1)\, a_1 \ln|y_1| + (\text{sgn } y_2) a_2 \ln|y_2| + \ldots + (\text{sgn } y_n)\, a_n \ln|y_n| = 0, \quad (9)$$

where:

$$(\text{sgn } y_1)\, a_1 + (\text{sgn } y_2)\, a_2 + \ldots + (\text{sgn } y_n)\, a_n = 0,$$

as before. For this frontier to have proper curvature there is only one output; [12] the elasticity of substitution between inputs is constant and equal to unity. We conclude that a commodity-wise additive and homogeneous production possibility frontier is unsuitable for representation of production possibilities with several outputs and several inputs.

3) *Group-wise Additivity*

As an extension of our characterization of a production possibility frontier with constant returns to scale, we introduce the concept of additivity in commodity groups. The production possibility frontier is *group-wise additive* in m mutually exclusive and exhaustive commodity groups if and only if the frontier can be represented in the form:

$$F(y_1, y_2 \ldots y_n) = F^1(y_1 \ldots y_{n_1}) + F^2(y_{n_1+1} \ldots y_{n_1+n_2}) + \ldots + F^m(y_{n_1+n_2+\ldots n_{m-1}+1} \ldots y_{n_1+n_2+\ldots+n_m}) = 0, \quad (10)$$

where $\Sigma n_i = n$, the number of commodities. For commodity-wise additivity each group consists of a single commodity and the number of groups is the same as the number of commodities.

The production possibility frontier is homogeneous and group-wise additive if and only if

[9] The concept of commodity-wise additivity is the same as the concept of additive or strong separability employed by Goldman and Uzawa (1964).

[10] This proposition was first derived in the theory of consumer behavior by Bergson (1936); Samuelson (1965) pointed out the second part of the proposition. Similar results had been obtained earlier in the literature on mean value functions; see Hardy, Littlewood, and Polya (1959), pp. 65–69, and the references given there.

[11] In this representation and those that follow, we assume that dimensions of the net outputs are chosen so that the coefficients $\{(\text{sgn } y_i)a_i\}$ sum to zero.

[12] See Mundlak (1964).

(1) The functions $[F^i]$ are homogeneous of the same degree, or

(2) The functions $[F^i]$ are logarithmic functions of functions homogeneous of degree one.

If the functions $[F^i]$ are homogeneous of the same degree, the production possibility frontier can be represented in the form:

$$F(y_1, y_2 \ldots y_n) = (\operatorname{sgn} G^1)\, a_1 |G^1|^\rho + (\operatorname{sgn} G^2)\, a_2 |G^2|^\rho + \ldots + (\operatorname{sgn} G^m)\, a_m |G^m|^\rho = 0. \quad (11)$$

Alternatively, if the functions $[F^i]$ are logarithmic, the production possibility frontier can be represented in the form:

$$F(y_1, y_2 \ldots y_n) = (\operatorname{sgn} G^1)\, a_1 \ln|G^1| + (\operatorname{sgn} G^2)\, a_2 \ln|G^2| + \ldots + (\operatorname{sgn} G^m)\, a_m \ln|G^m| = 0. \quad (12)$$

In both representations the functions $[G^i(y_{n_1+n_2+\cdots+n_{i-1}+1} \cdots y_{n_1+n_2+\cdots+n_i})]$ are homogeneous of degree one, $a_i > 0$ $(i = 1, 2 \ldots m)$, and:

$$(\operatorname{sgn} G^1)\, a_1 + (\operatorname{sgn} G^2)\, a_2 + \ldots + (\operatorname{sgn} G^m)\, a_m = 0.$$

A homogeneous and group-wise additive production possibility frontier has only one group of outputs (logarithmic or homogeneous with $\rho < 1$) or only one group of inputs $(\rho > 1)$, unless the commodity groups are perfect substitutes $(\rho = 1)$. In any case the production possibility frontier is additive in inputs and in outputs, considered as commodity groups. We conclude that any test of the implications of additivity should begin with a test of additivity between inputs and outputs.

We have defined additivity for individual commodities and for commodity groups. We have obtained explicit representations for production possibility frontiers characterized by homogeneity and commodity-wise or group-wise additivity. We can extend our characterization by defining *two-level additivity* as group-wise additivity together with commodity-wise additivity for each commodity group. Two-level additivity is not equivalent to commodity-wise additivity for the production possibility frontier as a whole. We can further extend our characterization by extending two-level additivity to any number of levels. Homogeneity and multi-level additivity imply that the production possibility frontier can be represented by means of compositions of power functions and logarithmic functions. To illustrate the representation of additive and homogeneous production possibility frontiers we consider the following examples:

(1) For one output and two inputs a homogeneous and commodity-wise additive production possibility frontier can be represented in the form:

$$y_1^\rho = a|y_2|^\rho + (1 - a)\, |y_3|^\rho, \quad (13)$$

where y_1 is the level of output and $[|y_2|, |y_3|]$ are the levels of input or, alternatively, in the form:

$$\ln y_1 = a \ln|y_2| + (1 - a) \ln|y_3|. \quad (14)$$

The first representation is the constant elasticity of substitution (CES) production function introduced by Arrow, Chenery, Minhas, and Solow. The second representation is the Cobb-Douglas production function; the elasticity of substitution is constant and equal to unity.[13]

(2) For two outputs $[y_1, y_2]$ and two inputs $[|y_3|, |y_4|]$, a homogeneous production possibility frontier that is group-wise additive in the outputs and inputs, where each group is commodity-wise additive, can be represented in the form:

$$[a_1 y_1^{\rho_1} + (1 - a_1)\, y_2^{\rho_1}]^{1/\rho_1} = [a_2 |y_3|^{\rho_2} + (1 - a_2)\, |y_4|^{\rho_2}]^{1/\rho_2}, \quad (15)$$

or, alternatively, in the form:

$$\ln\, [a_1 y_1^{\rho_1} + (1 - a_1)\, y_2^{\rho_1}]^{1/\rho_1} = a_2 \ln|y_3| + (1 - a_2) \ln|y_4|. \quad (16)$$

The first representation has constant elasticity of transformation (CET) between the two outputs and constant elasticity of substitution (CES) between the two inputs. In the second the elasticity of substitution is equal to unity. The CET-CES representation was introduced by Powell and Gruen.[14]

(3) For one output y_1 and four inputs $[|y_2|, |y_3|, |y_4|, |y_5|]$ a homogeneous production possibility frontier that is group-wise additive in output, the first two inputs $[|y_2|, |y_3|]$ and the second two inputs $[|y_4|, |y_5|]$, where each group

[13] See Arrow, Chenery, Minhas, and Solow (1961) and Douglas (1948).

[14] See Powell and Gruen (1968).

is commodity-wise additive, can be represented in the form:

$$y_1^{\rho_1} = a_1|G^2|^{\rho_1} + (1 - a_1)\,|G^3|^{\rho_1}, \qquad (17)$$

where:

$$-G^2 = [a_2|y_2|^{\rho_2} + (1 - a_2)\,|y_3|^{\rho_2}]^{1/\rho_2},$$

$$-G^3 = [a_3|y_4|^{\rho_3} + (1 - a_3)\,|y_5|^{\rho_3}]^{1/\rho_3};$$

or, alternatively:

$$\ln|G^2| = \frac{1}{2}\ln|y_2| + \frac{1}{2}\ln|y_3|;$$

$$\ln|G^3| = \frac{1}{2}\ln|y_4| + \frac{1}{2}\ln|y_5|;$$

or, finally:

$$\ln y_1 = a_1 \ln|G^2| + (1 - a_1)\ln|G^3|, \qquad (18)$$

with either form of the functions G^2 and G^3 given above. The first representation is the two-level constant elasticity of substitution production function introduced by Sato. The second representation was introduced by McFadden and the third by Uzawa.[15]

4) *Price Possibility Frontier*

Under constant returns to scale the level of profit associated with any set of prices is either zero or positively infinite. We define the set of price possibilities as the set of prices for which profit is equal to zero. We define the price possibility frontier as the frontier of the set of price possibilities. By duality in the theory of production we can characterize the production possibility frontier in terms of the price possibility frontier.[16] We first represent the price possibility frontier in the form:

$$P\,(q_1, q_2 \ldots q_n) = 0, \qquad (19)$$

where P is the level of profit associated with the set of prices $[q_i]$.

The price possibility frontier is homogeneous, so that:

$$P\,(\lambda q_1, \lambda q_2 \ldots \lambda q_n) = P\,(q_1, q_2 \ldots q_n) = 0, \qquad (20)$$

for any $\lambda > 0$. The production possibility frontier and the necessary conditions for producer equilibrium are dual to the price possibility frontier and its derivatives

$$\frac{y_i}{y_j} = \frac{\partial P/\partial q_i}{\partial P/\partial q_j}, \qquad (i \neq j;\ i, j = 1, 2 \ldots n). \qquad (21)$$

Derivatives of the price possibility frontier, holding the level of profit constant and equal to zero, are equal to the relative product and factor intensities.

Under constant returns to scale, if the production possibility frontier is commodity-wise additive, then the price possibility frontier is commodity-wise additive[17] and can be represented in the form

$$P\,(q_1, q_2 \ldots q_n) = P^1(q_1) + P^2(q_2) + \ldots + P^n(q_n) = 0, \qquad (22)$$

where the functions $[P^i]$ are strictly monotone, depend on only a single variable, and

(1) The functions $[P^i]$ are homogeneous of the same degree, or

(2) The functions $[P^i]$ are logarithmic.

If the functions $[F^i]$ in the representation of the production possibility frontier are homogeneous of degree ρ, the functions $[P^i]$ in the representation of the corresponding price possibility frontier are homogeneous of degree η,[18] where:

$$\frac{1}{\rho} + \frac{1}{\eta} = 1. \qquad (23)$$

The functions $[F^i]$ can be represented in the form:

$$F^i(y_i) = (\operatorname{sgn} y_i)\, a_i\, |y_i|^{\rho}, \qquad (i = 1, 2 \ldots n), \qquad (24)$$

and the functions $[P^i]$ can be represented in the form:

$$P^i(q_i) = (\operatorname{sgn} y_i)\, b_i q_i^{\eta}, \qquad (i = 1, 2 \ldots n), \qquad (25)$$

where:

$$b_i = a_i^{1-\eta}, \qquad (i = 1, 2 \ldots n). \qquad (26)$$

Alternatively, if the functions $[F^i]$ in the representation of the production possibility frontier are logarithmic, the functions $[P^i]$ in the representation of the price possibility frontier are logarithmic and we can write:

[15] See Sato (1967), McFadden (1963), and Uzawa (1962). The relationship between homogeneity and additivity and constancy of the elasticity of substitution is discussed in greater detail by Berndt and Christensen (1973a).

[16] See the references given above in footnote 5. Duality in production was first discussed by Hotelling (1932). Duality is also discussed by Diewert (1973), Gorman (1968), Jorgenson and Lau (1973), Lau (1972), McFadden (1973), and Shephard (1970).

[17] An analogous result was obtained by Lau (1972).

[18] For the duals of power functions and logarithmic functions, see Rockafellar (1970), pp. 105–107.

$$P\,(q_1, q_2 \ldots q_n) = (\operatorname{sgn} y_1)\, b_1 \ln q_1 + (\operatorname{sgn} y_2)\, b_2 \ln q_2 + \ldots + (\operatorname{sgn} y_n)\, b_n \ln q_n = 0, \quad (27)$$

where:

$$b_i = a_i \qquad (i = 1, 2 \ldots n), \quad (28)$$

in the logarithmic representation of the functions $[F^i]$ given above.

Under constant returns to scale, if the production possibility frontier is group-wise additive, the price possibility frontier is group-wise additive in the same commodity groups. We can extend our representations of the price possibility frontier to group-wise additivity and to multi-level additivity in an obvious way. Multi-level additive and homogeneous production possibility frontiers are self-dual in the sense of Houthakker.[19] For our examples of additive and homogeneous production possibility frontiers the corresponding price frontiers have the same functional form. The parameters of these price frontiers can be determined from the parameters of the corresponding production frontiers, but the two sets of parameters are not identical.

The first step in testing additivity is to test group-wise additivity between inputs and outputs, considered as commodity groups. Since a group-wise additive and homogeneous production possibility frontier corresponds to a group-wise additive price possibility frontier, the hypothesis of group-wise additivity of inputs and outputs can be tested directly by means of the production frontier or indirectly by means of the price possibility frontier. Similarly, since a multi-level additive and homogeneous production possibility frontier is self-dual, the hypothesis of multi-level additivity can be tested by means of either frontier.

III Transcendental Logarithmic Frontiers

1) *Introduction*

Our objective is to develop tests of the theory of production that do not employ additivity and homogeneity as part of the maintained hypothesis. For this purpose we introduce new representations of the production possibility frontier and the price possibility frontier. We refer to our representation of the production possibility frontier as the transcendental logarithmic produotion possibility frontier. Similarly, we refer to representation of the price possibility frontier as the transcendental logarithmic price possibility frontier. Each frontier is a transcendental logarithmic function in its arguments, the logarithms of quantities and prices, respectively. More simply, we refer to our production frontier and price frontier as the *translog production and price frontiers.*

2) *Translog Production Frontier*

In presenting the translog production frontier it is useful to specialize to the case of two outputs and two inputs. The basic approach is easily extended to any number of outputs and inputs. We suppose that there are two outputs — consumption C and investment I — and two inputs — capital K and labor L. The corresponding prices are q_C, q_I, q_K, q_L. The production possibility frontier F can be represented in the form:

$$F(C, I, K, L, A) = 0, \quad (29)$$

where A is an index of technology.

We approximate the logarithm of the production frontier plus unity by a function of the logarithms of the outputs and inputs,[20]

$$\begin{aligned} \ln (F + 1) = {} & \alpha_0 + \alpha_C \ln C + \alpha_I \ln I + \alpha_K \ln K + \alpha_L \ln L \\ & + \alpha_A \ln A + \ln C\,(\tfrac{1}{2} \beta_{CC} \ln C + \beta_{CI} \ln I + \beta_{CK} \ln K + \beta_{CL} \ln L + \beta_{CA} \ln A \\ & + \ln I\,(\tfrac{1}{2} \beta_{II} \ln I + \beta_{IK} \ln K + \beta_{IL} \ln L + \beta_{IA} \ln A) \\ & + \ln K\,(\tfrac{1}{2} \beta_{KK} \ln K + \beta_{KL} \ln L + \beta_{KA} \ln A) + \ln L\,(\tfrac{1}{2} \beta_{LL} \ln L + \beta_{LA} \ln A) \\ & + \ln A\,(\tfrac{1}{2} \beta_{AA} \ln A). \end{aligned} \quad (30)$$

The implications of the theory of production are invariant with respect to transformations of the production possibility frontier equal to

[19] See Houthakker (1965); self-duality is also discussed by Samuelson (1965).

[20] For convenience we have adopted the convention that outputs and inputs are measured as non-negative quantities.

zero when the frontier is equal to zero.[21] As one example of such a transformation, we add unity and take logarithms. More generally, we can transform the production possibility frontier, obtaining $\phi[\ln(F+1)]$ as the new frontier, where $\phi(0)$ is equal to zero. We obtain the following coefficients of the translog approximation at $\ln C = \ln I = \ln K = \ln L = \ln A = 0$:

$$\phi = \alpha_0,$$
$$\phi' \frac{\partial \ln F}{\partial \ln C} = \alpha_C ,$$
$$\phi' \frac{\partial^2 \ln F}{\partial \ln C^2} + \phi'' \frac{\partial \ln F}{\partial \ln C} \frac{\partial \ln F}{\partial \ln C} = \beta_{CC}. \quad (31)$$

The function ϕ is equal to zero for F equal to zero, but is otherwise arbitrary. We can choose ϕ' and ϕ'' for convenience in representing the translog approximation to an arbitrary production possibility frontier. A convenient normalization is

$$\alpha_K + \alpha_L = -1, \ \beta_{CK} + \beta_{CL} + \beta_{IK} + \beta_{IL} = 0. \quad (32)$$

The first normalization, $\alpha_K + \alpha_L = -1$, is required for estimation of the parameters of equations for the value ratios. The second normalization has the convenient property that a translog approximation to a production possibility frontier group-wise additive in outputs and inputs is group-wise additive. Given this normalization, we can determine one of the four parameters $[\beta_{CK}, \beta_{CL}, \beta_{IK}, \beta_{IL}]$ from the remaining three.

Using the translog form for the production possibility frontier and the necessary conditions for producer equilibrium, we obtain the ratio of the value of investment goods output to the value of capital input:

$$\frac{q_I I}{q_K K} = -\frac{\psi_I}{\psi_K}, \quad (33)$$

where $\psi_I = \alpha_I + \beta_{CI} \ln C + \beta_{II} \ln I + \beta_{IK} \ln K + \beta_{IL} \ln L + \beta_{IA} \ln A$

and ψ_K is similarly defined.

Similarly, the ratio of the value of labor input to the value of capital input is

$$\frac{q_L L}{q_K K} = -\frac{\psi_L}{\psi_K}, \quad (34)$$

where ψ_L is an expression similar to ψ_I and ψ_K.

At this point we specialize the discussion to national accounting data for which the value of output is equal to the value of input,

$$q_C C + q_I I = q_K K + q_L L. \quad (35)$$

Given any two of the value ratios in the expression,

$$\frac{q_C C}{q_K K} + \frac{q_I I}{q_K K} = 1 + \frac{q_L L}{q_K K},$$

the third is determined by the accounting identity. This implies that the parameters of the equation for the ratio of the value of consumption goods output to the value of capital input,

$$\frac{q_C C}{q_K K} = -\frac{\psi_C}{\psi_K}, \quad (36)$$

where ψ_C is also similar to ψ_I and ψ_K,

can be determined from those for the remaining two ratios. In fact,

$$\begin{aligned}
\alpha_C + \alpha_I + \alpha_K + \alpha_L &= 0\\
\beta_{CC} + \beta_{CI} + \beta_{CK} + \beta_{CL} &= 0\\
\beta_{CI} + \beta_{II} + \beta_{IK} + \beta_{IL} &= 0\\
\beta_{CK} + \beta_{IK} + \beta_{KK} + \beta_{KL} &= 0\\
\beta_{CL} + \beta_{IL} + \beta_{KL} + \beta_{LL} &= 0\\
\beta_{CA} + \beta_{IA} + \beta_{KA} + \beta_{LA} &= 0.
\end{aligned}$$

3) *Translog Price Frontier*

The translog price possibility frontier can be presented in the same way as the translog production possibility frontier. The price possibility frontier P can be represented in the form:

$$P\,(q_C, q_I, q_K, q_L, A) = 0, \quad (37)$$

where A is the index of technology. We approximate the price frontier by a function quadratic in the logarithms,

$$\begin{aligned}
\ln (P+1) = {} & \alpha_0 + \alpha_C \ln q_C + \alpha_I \ln q_I \\
& + \alpha_K \ln q_K + \alpha_L \ln q_L + \alpha_A \ln A \\
& + \ln q_C \,(\tfrac{1}{2} \beta_{CC} \ln q_C + \beta_{CI} \ln q_I \\
& + \beta_{CK} \ln q_K + \beta_{CL} \ln q_L \\
& + \beta_{CA} \ln A) \\
& + \ln q_I \,(\tfrac{1}{2} \beta_{II} \ln q_I + \beta_{IK} \ln q_K \\
& + \beta_{IL} \ln q_L + \beta_{IA} \ln A) \\
& + \ln q_K \,(\tfrac{1}{2} \beta_{KK} \ln q_K \\
& + \beta_{KL} \ln q_L + \beta_{KA} \ln A) \\
& + \ln q_L \,(\tfrac{1}{2} \beta_{LL} \ln q_L \\
& + \beta_{LA} \ln A) \\
& + \ln A \,(\tfrac{1}{2} \beta_{AA} \ln A). \quad (38)
\end{aligned}$$

[21] See Hicks (1946).

As before, we can normalize the parameters of the price possibility frontier so that:

$$\alpha_K + \alpha_L = -1,$$
$$\beta_{CK} + \beta_{CL} + \beta_{IK} + \beta_{IL} = 0. \tag{39}$$

Differentiating the translog price frontier, while holding the level of profit at zero, we obtain the relative net supply functions; for example, the ratio of the value of investment goods to the value of capital services is

$$\frac{q_I I}{q_K K} = -\frac{\psi^*_I}{\psi^*_K}, \tag{40}$$

where $\psi^*_I = \alpha_I + \beta_{CI} \ln q_C + \beta_{II} \ln q_I + \beta_{IK} \ln q_K + \beta_{IL} \ln q_L + \beta_{IA} \ln A$ and ψ^*_K is similarly defined.

The form of the functions determining the value ratios is identical to that for the translog production possibility frontier with prices in place of quantities. Given any two of the value ratios, the third is determined by the accounting identity between the value of output and the value of input.

IV Testing the Theory of Production

1) *Stochastic Specification*

The first step in implementing an econometric model of production based on the translog production frontier is to add a stochastic specification to the theoretical model based on the equations for the marginal rates of substitution. We add a disturbance term to each of the equations for the ratios of values of consumption goods output, investment goods output and labor input to the value of capital input. From the accounting identity relating these ratios we observe that the disturbance in any one of these equations can be determined from the disturbances in the remaining two. Only two of the three equations are required for a complete econometric model of production. Estimates of the parameters of the remaining equation are implied in the relationships among the parameters given above.

2) *Equality and Symmetry*

We estimate equations for ratios of the values of investment goods output and labor input to the value of capital input, subject to the normalization $\alpha_K + \alpha_L = -1$. If the equations are generated by profit maximization, the six parameters $[\alpha_K, \beta_{CK}, \beta_{IK}, \beta_{KK}, \beta_{KL}, \beta_{KA}]$ are the same for both equations. This results in a set of restrictions relating the parameters occurring in both equations. We refer to these as *equality restrictions.*

The production possibility frontier is twice differentiable, so that the Hessian of this frontier is symmetric. This gives rise to a set of restrictions relating the parameters of the cross-partial derivatives. For example, the parameter β_{IK} associated with $\ln K$ in the expression for $\partial F/\partial I$ must be equal to the same parameter associated with $\ln I$ in the expression for three parameters represented explicitly $[\beta_{IK}, \beta_{IL}, \beta_{KL}]$ and three additional parameters entering through the accounting identity between the value of output and the value of input $[\beta_{CI}, \beta_{CK}, \beta_{CL}]$. We refer to these as *symmetry restrictions.*

Constant returns to scale implies that the production possibility frontier satisfies:

$$F(\lambda C, \lambda I, \lambda K, \lambda L, A) = 0, \tag{41}$$

for any $\lambda > 0$. This implies the following restrictions on the parameters:

$$\begin{aligned}
&\alpha_C + \alpha_I + \alpha_K + \alpha_L = 0\\
&\beta_{CC} + \beta_{CI} + \beta_{CK} + \beta_{CL} = 0\\
&\beta_{CI} + \beta_{II} + \beta_{IK} + \beta_{IL} = 0\\
&\beta_{CK} + \beta_{IL} + \beta_{KK} + \beta_{KL} = 0\\
&\beta_{CL} + \beta_{IL} + \beta_{KL} + \beta_{LL} = 0\\
&\beta_{CA} + \beta_{IA} + \beta_{KA} + \beta_{LA} = 0\\
&\quad \frac{1}{2}\beta_{CC} + \beta_{CI} + \beta_{CK} + \beta_{CL}\\
&+ \frac{1}{2}\beta_{II} + \beta_{IK} + \beta_{IL}\\
&+ \frac{1}{2}\beta_{KK} + \beta_{KL} + \frac{1}{2}\beta_{LL} = 0.
\end{aligned}$$

Given symmetry restrictions on the six parameters $[\beta_{IK}, \beta_{IL}, \beta_{KL}, \beta_{CI}, \beta_{CK}, \beta_{CL}]$, the first six restrictions are identical to those derived from the accounting identity between the value of output and the value of input. The last restriction is implied in the second through fifth restrictions. We conclude that tests of the symmetry restrictions can also be interpreted as tests of constant returns to scale or homogeneity. Under the accounting identity between the value of output and the value of input, symmetry and homogeneity have precisely the same implications for the parameters of the functions determining the ratios of values of investment goods output and labor input to the value of capital input.

The parameters of the translog production possibility frontier can be normalized; only the normalization, $\beta_{CK} + \beta_{CL} + \beta_{IK} + \beta_{IL} = 0$, imposes a restriction on the parameters of equations for the value ratios. We refer to this as the *normalization restriction.* If equations for the value ratios are generated by profit maximization, subject to the translog production possibility frontier, the parameters satisfy equality, symmetry, and normalization restrictions.

3) Factor Augmentation

In the tests of the theory of production we employ the index of total factor productivity as a measure of the technology index A. The index of total factor productivity is invariant and path independent if and only if technical change can be represented by a single index.[22] If technical change is factor-augmenting and depends on a single index, we can write the production possibility frontier in the form:

$$F\,(C, I, K, L, A) = F\,(C, I, A \cdot K, A \cdot L) = 0. \qquad (42)$$

Furthermore, the index of technology A can be taken to be the index of total factor productivity,[23] implying the restrictions:

$$\begin{aligned}
\alpha_A &= \alpha_K + \alpha_L \\
\beta_{CA} &= \beta_{CK} + \beta_{CL} \\
\beta_{IA} &= \beta_{IK} + \beta_{IL} \\
\beta_{KA} &= \beta_{KK} + \beta_{KL} \\
\beta_{LA} &= \beta_{KL} + \beta_{LL} \\
\beta_{AA} &= \beta_{KK} + 2\beta_{KL} + \beta_{LL}.
\end{aligned}$$

The first and last of these restrictions can be employed in estimating the parameters $[\alpha_A, \beta_{AA}]$. Under the normalization suggested above:

$$\alpha_A = \alpha_K + \alpha_L = -1.$$

Give symmetry, the second restriction is implied by the third, fouth and fifth. We refer to the latter three as *factor augmentation restrictions.*

4) Group-wise Additivity

Group-wise additivity of the production possibility frontier in the two groups consisting of outputs and inputs implies that the frontier can be represented in the form:

$$Y\,(C, I, A) = X\,(K, L, A). \qquad (43)$$

Constant returns to scale implies that the functions $[Y(C, I, A), X(K, L, A)]$ can be taken to be homogeneous of degree one in outputs and inputs, respectively.

Under our normalization necessary and sufficient conditions for the translog production possibility frontier to be group-wise additive in the inputs and outputs are the following:

$$\beta_{CK} = \beta_{CL} = \beta_{IK} = \beta_{IL} = 0.$$

We refer to these as *group-wise additivity restrictions.* Only three of these restrictions are independent.[24]

If the production possibility frontier is group-wise additive in inputs and outputs and technical change is factor-augmenting we can write,

$$Y\,(C, I) = X\,(A \cdot K, A \cdot L). \qquad (44)$$

Constant returns to scale implies that we can write,

$$Y\,(C, I) = AX\,(K, L), \qquad (45)$$

where $Y(C, I)$ is an index of aggregate output, $X(K, L)$ is an index of aggregate input, and the functions $[Y(C, I), X(K, L)]$ are homogeneous of degree one.

5) Approximation

We have demonstrated that symmetry restrictions can also be interpreted as conditions for homogeneity under the accounting identity between the value of output and the value of input. Similarly, we can provide an alternative interpretation of the factor augmentation and group-wise additivity restrictions by considering the translog approximation to the CET-CES production possibility frontier: [25]

$$[\delta_1 C^{\rho_1} + (1 - \delta_1)\, I^{\rho_1}]^{1/\rho_1} = \gamma A\, [\delta_2 K^{\rho_2} + (1 - \delta_2)\, L^{\rho_2}]^{1/\rho_2}. \qquad (46)$$

This frontier is group-wise additive in two

[22] This is an implication of Hulten's (1973) conditions for invariance and path independence of Divisia index numbers; see Hulten (1973).

[23] See Solow (1967).

[24] Alternative tests for group-wise additivity have been proposed for versions of the "generalized Leontief" production possibility frontier by Denny (1973) and by Hall (1973).

[25] See Powell and Gruen (1968).

mutually exclusive and exhaustive commodity groups and each group is commodity-wise additive in the two commodities that comprise the group. Of course, the frontier as a whole is not commodity-wise additive.

The translog approximation of the CET-CES frontier can be obtained from:

$$\frac{1}{\rho_1} \ln [\delta_1 \exp (\rho_1 \ln C) + (1 - \delta_1) \exp \rho_1 \ln I)] - \ln \gamma - \ln A - \frac{1}{\rho_2} \ln [\delta_2 (\exp \rho_2 \ln K) + (1 - \delta_2) \exp \rho_2 \ln L)] = 0. \quad (47)$$

The approximating translog frontier (around the point $C = I = K = L = 1$) is:

$$\begin{aligned} \ln (F + 1) = {} & \alpha_0 + \alpha_I \ln I + \alpha_K \ln K + \alpha_L \ln L \\ & + \ln C \left(\frac{1}{2} \beta_{CC} \ln C - \beta_{CC} \ln I\right) \\ & + \ln I \left(\frac{1}{2} \beta_{CC} \ln I\right) \\ & + \ln K \left(\frac{1}{2} \beta_{KK} \ln K - \beta_{KK} \ln L\right) \\ & + \ln L \left(\frac{1}{2} \beta_{KK} \ln L\right) \\ & - \ln A; \end{aligned} \quad (48)$$

in this approximation:

$$\alpha_C + \alpha_I + \alpha_K + \alpha_L = 0.$$

The parameters of the translog approximation are:

$$\begin{aligned} \alpha_0 &= -\ln \gamma \\ \alpha_C &= \delta_1 \\ \alpha_I &= 1 - \delta_1 \\ \alpha_K &= -\delta_2 \\ \alpha_L &= -(1 - \delta_2) \\ \beta_{CC} &= \delta_1 (1 - \delta_1) \rho_1 \\ \beta_{KK} &= -\delta_2 (1 - \delta_2) \rho_2. \end{aligned}$$

Following Arrow, Chenery, Minhas, and Solow,[26] the parameter α_0 is the *efficiency parameter*, the parameters $[\alpha_C, \alpha_I, \alpha_K, \alpha_L]$ are *distribution parameters*, and the parameters $[\beta_{CC}, \beta_{KK}]$ are *substitution parameters*. This interpretation can be carried over to an unrestricted translog frontier, adding substitution parameters $[\beta_{II}, \beta_{LL}, \beta_{CI}, \beta_{CK}, \beta_{CL}, \beta_{IK}, \beta_{IL}, \beta_{KL}]$. The CET-CES production possibility frontier restricts the substitution parameters to two. Under homogeneity, symmetry, and normalization restrictions the translog production possibility frontier involves five substitution parameters. Imposing group-wise additivity in inputs and outputs, the translog frontier involves only two substitution parameters.

A test of the factor augmentation and group-wise additivity restrictions can also be interpreted as a test of the CET-CES frontier or of the CES production function. In the CES production function the index of aggregate output $Y(C, I)$ is measured directly; this index is invariant and path independent if and only if the production possibility frontier is group-wise additive in inputs and outputs. This additivity condition is weaker than the additivity conditions underlying the CET-CES frontier, since the CET-CES frontier also implies commodity-wise additivity of aggregate output and aggregate input in the individual commodities.[27]

Tests of the CET-CES frontier and the CES production function involve a possible error of approximation if the true production possibility frontier is CET-CES or the true production function is CES. Under constant returns to scale and group-wise additivity between inputs and outputs the only translog production possibility frontier that is commodity-wise additive in capital and labor input involves an index of aggregate input that is Cobb-Douglas in form:[28]

$$\ln X = -\alpha_K \ln K + (1 + \alpha_K) \ln L, \quad (49)$$

so that

$$\beta_{KK} = 0.$$

A similar restriction for the index of aggregate output implies an output index that violates the convexity conditions for the production possibility frontier.

6) *Duality*

In implementing an econometric model of production based on the translog price possibility frontier the first step, as before, is to add a stochastic specification. Only two of the three equations for ratios of the values of con-

[26] Arrow, Chenery, Minhas and Solow (1961), p. 230.

[27] See Hulten (1973).

[28] See Douglas (1948). Berndt and Christensen (1973b, 1973c) have provided a detailed empirical analysis of the internal structure of aggregate input for United States manufacturing.

sumption goods output, investment goods output, and labor input to the value of capital input are required for a complete econometric model of production. As before, a convenient normalization is that $\alpha_K + \alpha_L = -1$. The restrictions on the translog price possibility frontier are strictly analogous to restrictions on the translog production possibility frontier. *Equality, normalization, and symmetry restrictions* are the same as before. Tests of the symmetry restrictions can also be interpreted as tests of homogeneity of the price possibility frontier.

Factor-augmenting technical change implies that the price possibility frontier can be written:

$$P\,(q_C, q_I, q_K, q_L, A) = P\,(q_C, q_I, q_K/A, q_L/A) = 0, \qquad (50)$$

where A can be taken to be the index of total factor productivity.[29] Factor augmentation implies the restrictions:

$$\begin{aligned} -\alpha_A &= \alpha_K + \alpha_L \\ -\beta_{CA} &= \beta_{CK} + \beta_{CL} \\ -\beta_{IA} &= \beta_{IK} + \beta_{IL} \\ -\beta_{KA} &= \beta_{KK} + \beta_{KL} \\ -\beta_{LA} &= \beta_{KL} + \beta_{LL} \\ \beta_{AA} &= \beta_{KK} + 2\beta_{KL} + \beta_{LL}. \end{aligned}$$

Group-wise additivity of the production possibility frontier in the two inputs, capital and labor, in the two outputs, consumption and investment, and homogeneity imply that the price possibility frontier can be written in the form:

$$q_Y\,(q_C, q_I, A) = q_X\,(q_K, q_L, A), \qquad (51)$$

where the functions $[q_Y(q_C, q_I, A), q_X(q_K, q_L, A)]$ can be taken to be homogeneous of degree one in the prices of outputs and the prices of inputs, respectively.

Under our normalization necessary and sufficient conditions for the translog price possibility frontier to be group-wise additive in the prices of outputs and inputs are:

$$\beta_{CK} = \beta_{CL} = \beta_{IK} = \beta_{IL}.$$

Only three of these restrictions are independent.

As before, the *factor augmentation and group-wise additivity restrictions* can also be interpreted as restrictions arising from the translog approximation of the CET-CES price possibility frontier. This frontier can be represented in the form:

$$A\,[\zeta_1 q_C^{\eta_1} + (1 - \zeta_1)\, q_I^{\eta_1}]^{1/\eta_1} = \theta\,[\zeta_2 q_K^{\eta_2} + (1 - \zeta_2)\, q_L^{\eta_2}]^{1/\eta_2} \qquad (52)$$

where:

$$\frac{1}{\rho_1} + \frac{1}{\eta_1} = \frac{1}{\rho_2} + \frac{1}{\eta_2} = 1,$$

and:

$$\frac{\zeta_1}{1-\zeta_1} = \left(\frac{\delta_1}{1-\delta_1}\right)^{1-\eta_1},$$

$$\frac{\zeta_2}{1-\zeta_2} = \left(\frac{\delta_2}{1-\delta_2}\right)^{1-\eta_2},$$

$$\theta = \frac{[\delta_2^{1-\eta_2} + (1-\delta_2)^{1-\eta_2}]^{1/\eta_2}}{\gamma\,[\delta_1^{1-\eta_1} + (1-\delta_1)^{1-\eta_1}]^{1/\eta_1}},$$

since the CET-CES production possibility frontier is self-dual, that is, the price possibility frontier has the same functional form.

Under group-wise additivity between output prices and input prices, commodity-wise additivity of the translog price possibility frontier in the prices of capital and labor inputs implies:

$$\beta_{KK} = 0,$$

as before.

The translog production possibility frontier and the translog price possibility frontier do not correspond to the same technology. However, these frontiers can be regarded as alternative approximations to the same underlying technology. If, for a given technology, both the production frontier and the price frontier can be represented in closed form, for example with CET-CES frontiers, the error of approximation can be assessed by measuring the discrepancy between the frontiers and their translog approximations.

V Empirical Results

1) *Summary of Tests*

Our objective has been to develop statistical tests of the theory of production that do not employ the assumptions of homogeneity and additivity. At this point it is useful to summarize the restrictions on the translog production possibility frontier corresponding to the

[29] Under constant returns, equal rates of factor augmentation may be interpreted equivalently as equal rates of decline of factor prices.

theory of production and to functional forms that are homogeneous and additive. Restrictions on the translog price possibility frontier are analogous. We present these restrictions in a form corresponding to the two equations for ratios of the values of investment output and labor input to the value of capital input:

(1) *Equality restrictions*: The parameters $[\alpha_K, \beta_{CK}, \beta_{IK}, \beta_{KK}, \beta_{KL}, \beta_{KA}]$ occur in both equations and must take the same value.

(2) *Symmetry restrictions*: The parameters $[\beta_{IK}, \beta_{IL}, \beta_{KL}, \beta_{CK}, \beta_{CL}, \beta_{CI}]$ are the same wherever they occur and must take the same value.

(3) *Normalization restriction*: $\beta_{CK} + \beta_{CL} + \beta_{IK} + \beta_{IL} = 0$.

(4) *Factor augmentation restrictions*: $\beta_{IA} = \beta_{IK} + \beta_{IL}$, $\beta_{KA} = \beta_{KK} + \beta_{KL}$, $\beta_{LA} = \beta_{KL} + \beta_{LL}$.

(5) *Group-wise additivity restrictions*: $\beta_{CK} = \beta_{CL} = \beta_{IK} = 0$.

(6) *Commodity-wise additivity in capital and labor input*: $\beta_{KK} = 0$.

The symmetry restrictions, given the identity between the value of output and the value of input, are equivalent to restrictions implied by homogeneity. The factor augmentation and group-wise additivity restrictions are equivalent to restrictions implied by the translog approximation to the CET-CES production possibility frontier. The commodity-wise additivity restriction in capital and labor input implies that aggregate input can be represented in Cobb-Douglas form. In addition to the restrictions on the parameters to be estimated from the two behavioral equations we employ the following restrictions to estimate the remaining parameters:

(1) *Normalization*: $\alpha_K + \alpha_L = -1$.

(2) *Constant returns*: $\alpha_C + \alpha_I + \alpha_K + \alpha_L = 0$, $\beta_{CA} + \beta_{IA} + \beta_{KA} + \beta_{LA} = 0$.

(3) *Factor augmentation*: $\alpha_A = \alpha_K + \alpha_L$, $\beta_{AA} = \beta_{KK} + 2\beta_{KL} + \beta_{LL}$.

2) *Estimation*

Our empirical results are based on time series data for the United States private domestic economy for 1929–1969.[30] We have fitted the parameters of the translog production possibility frontier, employing the stochastic specification outlined above. Under this specification there are two behavioral equations corresponding to ratios of the values of investment goods and labor services to the value of capital services. Similarly, we have fitted the parameters of the translog price possibility frontier, employing an analogous stochastic specification. The production and price frontiers correspond to two distinct representations of technology. There are forty-one observations for each behavioral equation, so that the number of degrees of freedom available for either direct or indirect statistical tests is eighty-two.

Our maintained hypothesis corresponds to the unrestricted form of the two behavioral equations derived from the production possibility frontier. The unrestricted behavioral equations, estimated under the normalization $\alpha_K + \alpha_L = -1$, involve twenty-two unknown parameters or eleven unknown parameters in each equation. Unrestricted estimates of these parameters are presented in the first column of table 1.[31] The first hypothesis to be tested is that the theory of production is valid; the theory of production implies equality, normalization, and symmetry restrictions on the parameters of the translog production possibility frontier. There are twelve equality, normalization, and symmetry restrictions, so that the theory of production implies that the twenty-two unknown parameters of the translog production possibility frontier can be expressed as functions of only ten. Restricted estimates of the parameters of the production possibility frontier, obtained by imposing the equality, normalization, and symmetry restrictions, are presented in the second column of table 1. Analogous estimates of the parameters of the price possibility frontier are presented in the first two columns of table 2.

Given the validity of the theory of production, the second hypothesis to be tested is that technical change is factor augmenting and the production possibility frontier is group-wise additive in outputs and inputs. There are six

[30] The data are based on the estimates of Christensen and Jorgenson (1969, 1970), extended to 1969.

[31] We employ an iterative version of the three-stage least squares estimator proposed by Zellner and Theil (1962). This estimator is asymptotically equivalent to the maximum likelihood estimator.

TABLE 1. — ESTIMATES OF THE PARAMETERS OF THE TRANSLOG PRODUCTION POSSIBILITY FRONTIER

Parameters	Unrestricted estimates [a]	Equality, Normalization and Symmetry [b]	Factor Augmentation and Group-wise Additivity [c]	Commodity-wise Additivity [d]
		Investment Equation		
α_I	.304	.305	.301	.301
β_{CI}	−1.11	−.413	−.196	−.194
β_{II}	−.406	−.138	.196	.194
β_{IK}	.491	.115	e	e
β_{IL}	1.29	.436	e	e
β_{IA}	1.89	.993	e	e
β_{CK}	.476	−.497	e	e
β_{IK}	.411	.115	e	e
β_{KK}	−.228	.254	−.0468	e
β_{KL}	−1.05	.129	.0468	e
β_{KA}	−1.41	−.267	e	e
		Labor Equation		
α_L	−.618	−.613	−.617	−.610
β_{CL}	.801	−.054	e	e
β_{IL}	−.683	.436	e	e
β_{KL}	−.614	.129	.0468	e
β_{LL}	.519	−.511	−.0468	e
β_{LA}	1.74	−1.32	e	e
β_{CK}	.443	−.497	e	e
β_{IK}	−.598	.115	e	e
β_{KK}	−.410	.254	−.0468	e
β_{KL}	.655	.129	.0468	e
β_{KA}	1.32	−.267	e	e

[a] Unrestricted estimates under the normalization $\alpha_K + \alpha_L = -1$.
[b] Restricted estimates, equality, normalization, and symmetry restrictions imposed.
[c] Restricted estimates, factor augmentation and group-wise additivity restrictions together with equality, normalization, and symmetry restrictions imposed.
[d] Restricted estimates, commodity-wise additivity of aggregate input, together with equality, factor augmentation, normalization, group-wise additivity, and symmetry restrictions imposed.
[e] Parameter value constrained to be equal to zero.

TABLE 2. — ESTIMATES OF THE PARAMETERS OF THE TRANSLOG PRICE POSSIBILITY FRONTIER

Parameters	Unrestricted Estimates [a]	Equality, Normalization, and Symmetry [b]	Factor Augmentation and group-wise Additivity [c]	Commodity-wise Additivity [d]
		Investment Equation		
α_I	.315	.305	.301	.303
β_{CI}	1.68	−.099	−.506	−.587
β_{II}	.781	.283	.506	.587
β_{IK}	−.235	.360	e	e
β_{IL}	−1.72	−.544	e	e
β_{IA}	1.42	.366	e	e
β_{CK}	−.435	.698	e	e
β_{IK}	−.235	.360	e	e
β_{KK}	−.659	−.809	.049	e
β_{KL}	.317	−.250	−.049	e
β_{KA}	1.06	.983	e	e
		Labor Equation		
α_L	−.628	−.622	−.613	−.610
β_{CL}	−6.43	−.515	e	e
β_{IL}	−4.21	−.544	e	e
β_{KL}	.317	−.250	−.049	e
β_{LL}	5.86	1.31	.049	e
β_{LA}	−12.1	−1.01	e	e
β_{CK}	.826	.698	e	e
β_{IK}	−.650	.360	e	e
β_{KK}	.774	.809	.049	e
β_{KL}	−.553	−.250	−.049	e
β_{KA}	−.670	.983	e	e

[a] Unrestricted estimates under the normalization $\alpha_K + \alpha_L = -1$.
[b] Restricted estimates, equality, normalization, and symmetry restrictions imposed.
[c] Restricted estimates, factor augmentation and group-wise additivity restrictions together with equality, normalization, and symmetry restrictions imposed.
[d] Restricted estimates, commodity-wise additivity of aggregate input, together with equality, factor augmentation, normalization, group-wise additivity, and symmetry restrictions imposed.
[e] Parameter value constrained to be equal to zero.

factor augmentation and group-wise additivity restrictions, so that the ten parameters of the production possibility frontier implied by the theory of production can be expressed as a function of four parameters — a distribution and substitution parameter for aggregate output and corresponding parameters for aggregate input. The efficiency parameter for the frontier determines the units of measurement for the index of factor productivity. Restricted estimates of the parameters of the production possibility frontier, obtained by imposing the factor augmentation and group-wise additivity restrictions together with the equality, normalization, and symmetry restrictions, are presented in the third column of table 1. Analogous estimates of the parameters of the price possibility frontier are presented in the third column of table 2. Finally, the third hypothesis to be tested is that production possibility frontier is commodity-wise additive in labor and capital input. This hypothesis implies one additional restriction on the four parameters of the frontier. Restricted estimates, imposing this additional restriction, are presented in the fourth column of table 1; analogous estimates for the price possibility frontier are presented in the fourth column of table 2.

3) *Test statistics*

To test the validity of the theory of production and of restrictions on the form of the production possibility frontier we employ a "nested" series of tests. At each stage in the series we calculate the change in the weighted sum of squared residuals resulting from restrictions imposed at that stage. We divide this change by the sum of squared residuals at the previous stage. Finally, we divide both numerator and denominator of this ratio by the appropriate number of degrees of freedom. The resulting test statistics is distributed, asymptotically, as $F(\nu_1, \nu_2)$, where ν_1 is the numerator degrees of freedom and ν_2 is the denominator degrees of freedom. Of course, each F-ratio is distributed, asymptotically, as chi-squared divided by the numerator degrees of freedom. These test statistics are asymptotically equivalent to likelihood ratio test statistics. Critical values of F and chi-squared employed in our tests are given in table 3.[32]

TABLE 3. — CRITICAL VALUES OF $F(\nu_1, \nu_2)$ and χ^2/ν_1

Degrees of Freedom		Level of Significance				
		.10	.05	.025	.01	.005
$\nu_1 = 12$	$F(12,60)$	1.66	1.92	2.17	2.50	2.74
$\nu_2 = 60$	$\chi^2/12$	1.55	1.75	1.94	2.18	2.36
$\nu_1 = 3$	$F(3,72)$	2.12	2.74	3.32	4.09	4.69
$\nu_2 = 72$	$\chi^2/3$	2.08	2.60	3.12	3.78	4.28
$\nu_1 = 6$	$F(6,72)$	1.86	2.23	2.61	3.08	3.45
$\nu_2 = 72$	$\chi^2/6$	1.77	2.10	2.41	2.80	3.09
$\nu_1 = 3$	$F(3,75)$	2.17	2.74	3.31	4.09	4.67
$\nu_2 = 75$	$\chi^2/3$	2.08	2.60	3.12	3.78	4.28
$\nu_1 = 1$	$F(1,78)$	2.78	3.98	5.22	7.00	8.39
$\nu_2 = 78$	$\chi^2/1$	2.71	3.84	5.02	6.63	7.88

To control the overall level of significance for each of our series of tests, direct and indirect, we set the overall level of significance for each series at 0.05. We then allocate the overall level of significance among the various stages in each series of tests. We first assign levels of significance of 0.02 to tests of the theory of production and 0.03 to tests of restrictions on functional form. Given a level of significance of 0.03 for the validity of restrictions on the functional form, we assign 0.02 to group-wise additivity and factor augmentation and 0.01 to commodity-wise additivity. We test group-wise additivity and factor augmentation, proceeding conditionally on the validity of the theory of production. Finally, we test commodity-wise additivity in the inputs, proceeding conditionally on the validity of the theory of production and the validity of group-wise additivity and factor augmentation restrictions on functional form. With the aid of the critical levels presented in table 3, the reader can evaluate the results of these tests for a range of alternative levels of significance and for alternative allocations of the overall level of significance among stages of the series of tests for either direct or indirect representations of the production possibility frontier.

[32] The values for chi-squared are taken from tables for F with $\nu_2 = \infty$ degrees of freedom.

Test statistics for both direct and indirect tests of the theory of production and of restrictions on functional form are given in table 4. At a level of significance of 0.02 we accept the hypothesis that restrictions implied by the theory of production are valid for either the direct or the indirect series of tests. The F-ratio for the direct test is 0.00; the analogous ratio for the indirect test is 1.83. Proceeding conditionally on the validity of the theory of production, we can test the validity of restrictions on functional form. These restrictions include factor augmentation and group-wise additivity restrictions. Either set of restrictions can be valid without the other. We test the two sets of restrictions individually; of course, the tests are not "nested" so that the sum of levels of significance for each of the two hypotheses considered separately is an upper bound for the level of significance of tests of the two hypotheses considered simultaneously. Setting the level of significance for each test at .01 we obtain an upper bound on the overall level of significance of .02. At these levels of significance we reject the hypothesis that restrictions implied by group-wise additivity are valid for either direct or indirect tests. We accept the hypothesis that restrictions implied by factor augmentation are valid for the direct test; however, we reject this hypothesis for the indirect test.

We have tested the validity of restrictions on the form of the production possibility frontier, proceeding conditionally on the validity of the theory of production. We have tested restrictions implied by factor augmentation and by group-wise additivity individually. An alternative to our test procedure is to test the validity of factor augmentation and group-wise additivity restrictions jointly. At a level of significance of 0.02 we would reject the joint hypothesis for either direct or indirect

TABLE 4. — F-RATIOS FOR DIRECT AND INDIRECT TESTS OF THE THEORY OF PRODUCTION AND OF RESTRICTIONS ON THE FORM OF THE PRODUCTION AND PRICE POSSIBILITY FRONTIERS

	Degrees of Freedom	Direct	Indirect
		Theory of Production	
Equality, normalization, and symmetry restrictions	(12,60)	0.00[a]	1.83
		Functional Form	
Factor augmentation and group-wise additivity	(6,72)	6.39	38.5
Commodity-wise additivity	(1,78)	16.8	12.2

[a] The estimated change in the sum of squared residuals is negative.

tests. The F-ratio for the direct test is 6.39; for the indirect test the ratio is 38.5. An additional alternative to our test procedure is to test factor augmentation and then to test group-wise additivity, conditional on factor augmentation. Another alternative is to reverse this procedure. We present test statistics required for each of these alternative procedures in table 5.

TABLE 5. — F-RATIOS FOR DIRECT AND INDIRECT TESTS OF FACTOR AUGMENTATION AND GROUP-WISE ADDITIVITY.

	Degrees of Freedom	Direct	Indirect
		Unconditional	
Factor augmentation	(3,72)	3.23	58.1
Group-wise additivity	(3,72)	8.20	9.12
		Conditional	
Factor augmentation, given group-wise additivity	(3,75)	6.39	55.4
Group-wise additivity, given factor augmentation	(3,75)	14.0	20.3

For completeness we present test statistics for the hypothesis of commodity-wise additivity in the two inputs, conditional on the validity of restrictions implied by the theory of production and of factor augmentation and group-wise additivity restrictions. The F-ratio for the direct test is 16.8 and the corresponding ratio for the indirect test is 12.2. Our results for both direct and indirect tests are consistent with the theory of production. Our results for the indirect tests are inconsistent with restrictions on functional form implied by factor augmentation and group-wise additivity; our results for the direct tests are consistent with restrictions implied by factor augmentation but inconsistent with restrictions implied by group-wise additivity.

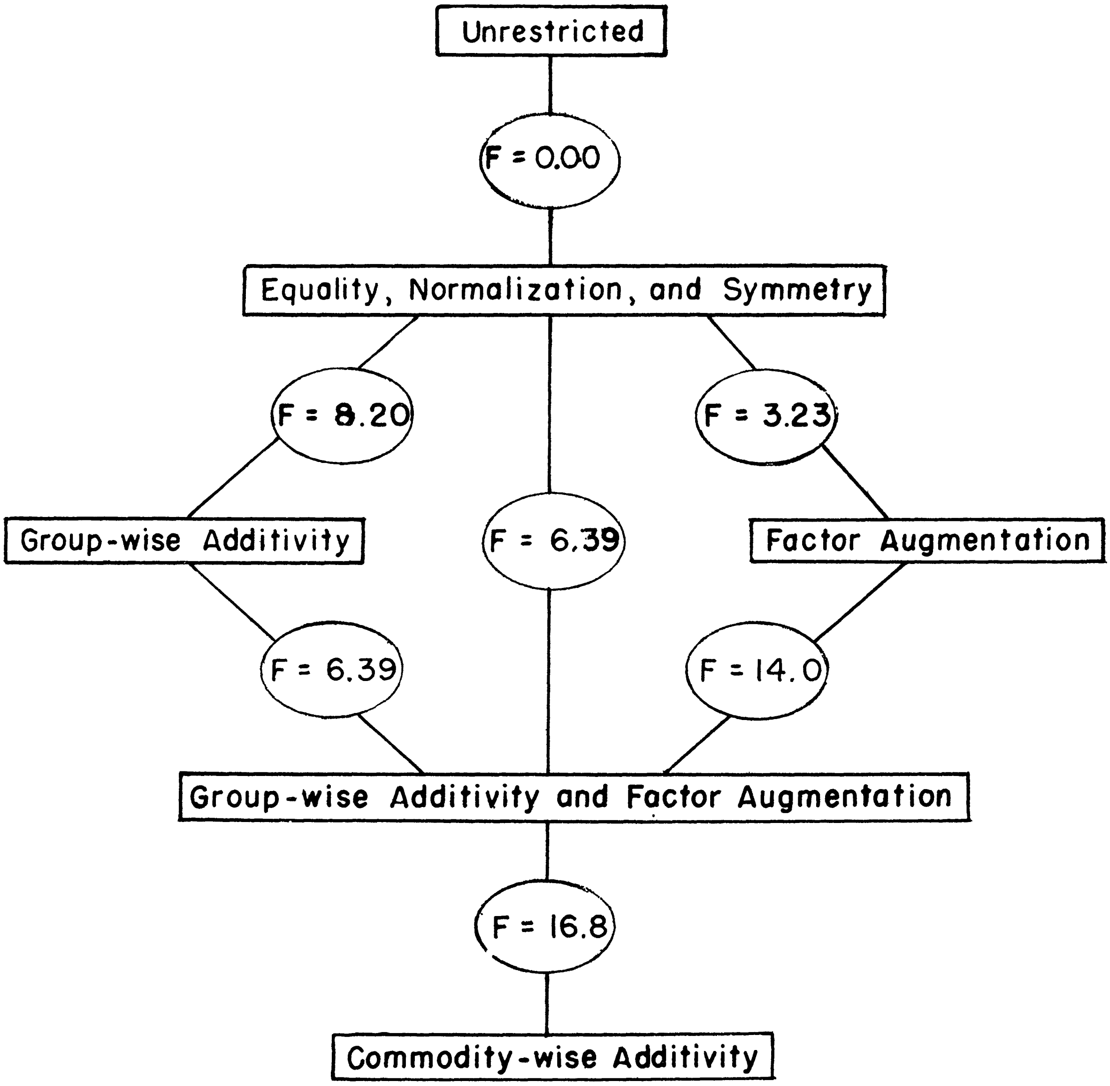
VI Summary and Conclusion

Our objective has been to test the theory of production without imposing the assumptions of additivity and homogeneity as part of the maintained hypothesis. We first examine the implications of these assumptions for the form of the production possibility frontier. We conclude that the assumption of commodity-wise additivity that underlies the constant elasticity of substitution production function is unsuitable as a basis for representing a production possibility frontier with several outputs and several inputs. Group-wise additivity implies that inputs and outputs, considered as commodity groups, must be additive. Multi-level additivity implies that the production possibility frontier is self-dual; the price possibility frontier has the same functional form with parameters that depend on the parameters of the production possibility frontier.

By duality in the theory of production the properties of the production possibility frontier and necessary conditions for producer equilibrium correspond to properties of the price possibility frontier and conditions for relative product and factor intensities. Duality is a natural tool for the construction of tests of the theory of production and tests of hypotheses about the form of the production possibility frontier, including hypotheses about additivity. The first step in testing additivity is to test group-wise additivity between inputs and outputs. This hypothesis can be tested directly by means of the transcendental logarithmic production possibility frontier or indirectly by means of the transcendental logarithmic price possibility frontier. Similarly, the hypothesis of multi-level additivity can be tested by means of either frontier.

Our empirical results are summarized in diagrammatic form in figures 1 and 2. For either direct or indirect series of tests we proceed from an unrestricted form of the behavioral equations to a form implied by equality, normalization, and symmetry restrictions. At this point we can proceed directly to

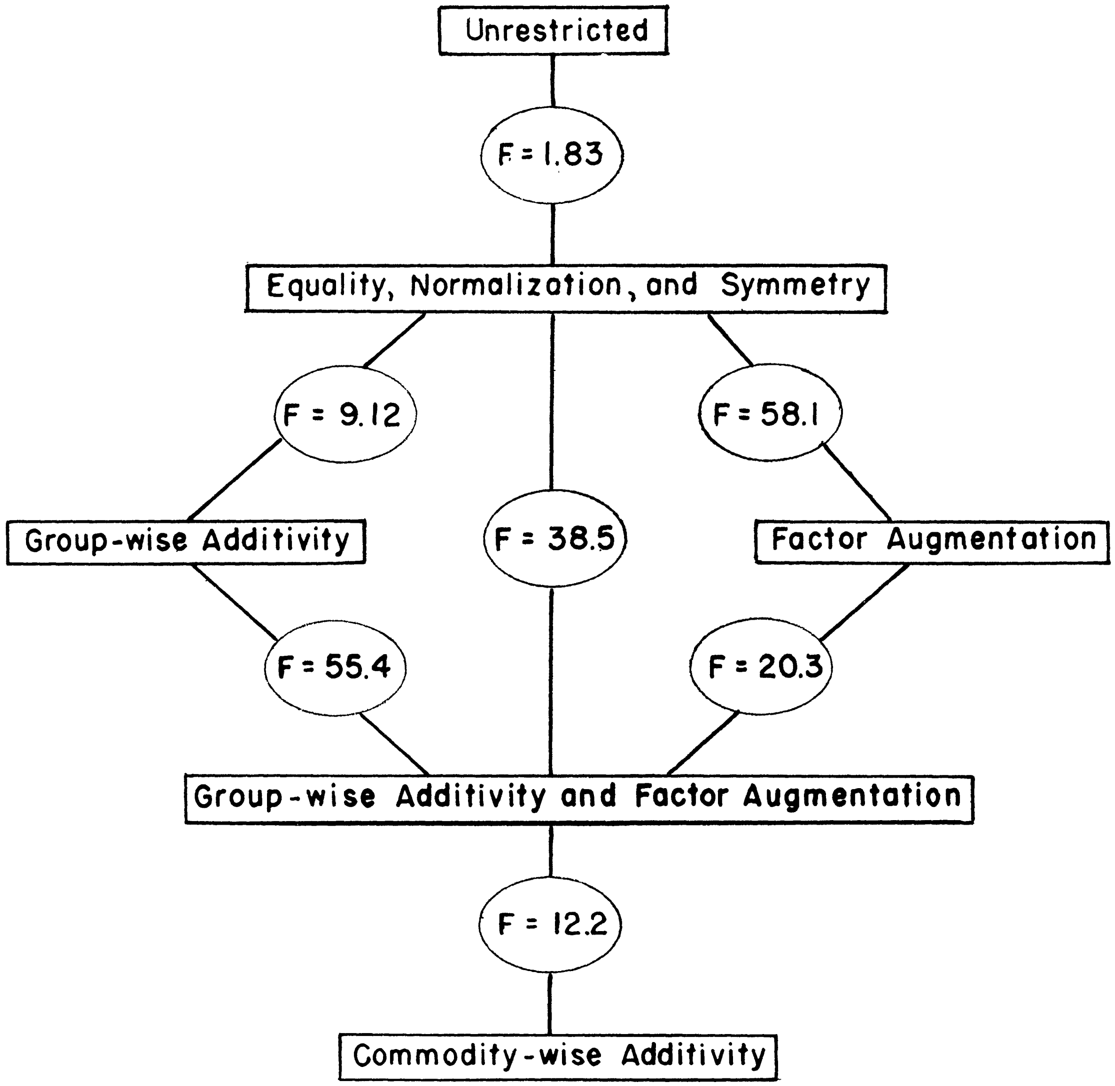
FIGURE 1. — DIRECT SERIES OF TESTS OF THE THEORY OF PRODUCTION AND THE FORM OF THE PRODUCTION POSSIBILITY FRONTIER



a form implied by separability and factor augmentation restrictions, together with equality, normalization, and symmetry, or we can proceed first to separability and then to factor augmentation or vice-versa. Finally, given equality, normalization, symmetry, separability and factor augmentation, we can proceed to a form implied by separability of aggregate input. F-ratios along each of these alternative paths of statistical inference are presented in figures 1 and 2 for the direct and indirect tests, respectively.

The results of our direct and indirect tests of the theory of production, based on the transcendental logarithmic production and price frontiers, are consistent with the validity of the extensive set of equality, symmetry, and normalization restrictions implied by the theory. Our results are inconsistent with the hypothesis of group-wise additivity of the production possibility frontier in inputs and outputs for either direct or indirect tests. Production possibility frontiers characterized by additivity and homogeneity have proved useful in representing production with one output and two inputs, as in the study of capital-labor sub-

FIGURE 2. — INDIRECT SERIES OF TESTS OF THE THEORY OF PRODUCTION AND THE FORM OF THE PRICE POSSIBILITY FRONTIER



stitution by Arrow, Chenery, Minhas, and Solow. The extension of this approach to production with two outputs and two inputs conflicts sharply with our empirical evidence.

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