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APPLICATIONS OF LORENZ CURVES IN ECONOMIC ANALYSIS

BY N. C. KAKWANI[1]

The Lorenz curve is widely used as a convenient graphical device to represent the size distribution of income and wealth. In the present paper the concept of the Lorenz curve has been extended and generalized to study the relationships among the distributions of different economic variables. Some theorems have been proved which have a large number of economic applications.

1. INTRODUCTION

THE LORENZ CURVE RELATES the cumulative proportion of income units to the cumulative proportion of income received when units are arranged in ascending order of their income. In the past the curve has been used mainly as a convenient graphical device to represent the size distribution of income and wealth.

The interest in the Lorenz curve technique has been revived recently by Atkinson [**1**] who provided a theorem relating the social welfare function and the Lorenz curve. He showed that the ranking of income distributions according to the Lorenz curve criterion is identical with the ranking implied by aggregate economic welfare regardless of the form of the welfare function of the individuals (except that it be increasing and concave) provided the Lorenz curves do not intersect. However, if the Lorenz curves do intersect, one can always find two functions that will rank them differently. Das Gupta, Sen, and Starrett [**2**] have shown that this result is in fact more general and does not depend on the assumption that the welfare functions should necessarily be additive.

In the present paper the Lorenz curve technique is used as a tool to introduce distributional considerations in economic analysis. The concept of the Lorenz curve has been extended and generalized to study the relationships among the distributions of different economic variables. The generalized Lorenz curves are called concentration curves and the Lorenz curve is only a special case of such curves, viz, the concentration curve for income.[2]

Section 2 gives the theorems and corrollaries relating the concentration curve of a function and its elasticity which provide the basis to study relationships among the distributions of different economic variables. Applications of the theorems are discussed in Section 3.

2. THE CONCENTRATION CURVES

Let x be the income and $F(x)$ be its distribution function which represents the proportion of income units having income less than or equal to x. If it is assumed

[1] This study was completed while the author was a staff member of the Development Research Center, World Bank. I wish to thank Graham Pyatt, Clive Bell, and two unknown referees for their comments on an earlier draft of this paper.

[2] Professor Mahalanobis [**8**] used concentration curves to describe the consumption pattern for different commodities based on the National Sample Survey data. See also Roy, Chakravarty, and Laha [**12**].

that the mean μ of the distribution exists, then the first moment distribution function $F_1(x)$ is defined and it represents the proportion of total income earned by income units having income less than or equal to x.

The Lorenz curve is the relationship between $F(x)$ and $F_1(x)$. The most widely used measure of income inequality is the Gini index which is equal to one minus twice the area under the Lorenz curve.

Let $g(x)$ be a continuous function of x such that its first derivative exists and $g(x) \geqslant 0$. If the mean $E[g(x)]$ exists, then one can define

$$F_1[g(x)] = \frac{1}{E[g(x)]} \int_0^x g(x) f(x)\, dx \tag{2.1}$$

where $f(x)$ is the probability density function of x so that $F_1[g(x)]$ is monotonic increasing and $F_1[g(0)] = 0$ and $F_1[g(\infty)] = 1$. The relationship between $F_1[g(x)]$ and $F(x)$ will be called the concentration curve of the function $g(x)$. It can be seen that the Lorenz curve of income x is a special case of the concentration curve for the function $g(x)$ when $g(x) = x$. Let $g^*(x)$ be another continuous function of x; then the graph of $F_1[g(x)] vs F_1[g^*(x)]$ will be called the relative concentration curve of $g(x)$ with respect to $g^*(x)$. Denote by $\eta_g(x)$ and $\eta_{g^*}(x)$ the elasticities of $g(x)$ and $g^*(x)$ with respect to x, respectively; then we can state the following theorem.

THEOREM 1: *The concentration curve for the function $g(x)$ will lie above (below) the concentration curve for the function $g^*(x)$ if $\eta_g(x)$ is less (greater) than $\eta_{g^*}(x)$ for all $x \geqslant 0$.*

PROOF OF THEOREM 1: Equation (2.1) gives the slope of the relative concentration curve of $g(x)$ with respect to $g^*(x)$ as

$$\frac{dF_1[g(x)]}{dF_1[g^*(x)]} = \frac{E[g^*(x)]g(x)}{E[g(x)]g^*(x)} \tag{2.2}$$

which implies that the relative concentration curve is monotonic increasing. Since the curve must pass through (0, 0) and (1, 1), it follows that a sufficient condition for $F_1[g(x)]$ to be greater (less) than $F_1[g^*(x)]$ is that the curve be convex (concave) from above. To establish curvature we obtain the second derivative of $F_1[g(x)]$ with respect to $F_1[g^*(x)]$ as:

$$\frac{d^2F_1[g(x)]}{dF_1^2[g^*(x)]} = \frac{(E[g^*(x)])^2}{E[g(x)]} \frac{g(x)}{g^{*2}(x)} \frac{[\eta_g - \eta_{g^*}]}{xf(x)}, \tag{2.3}$$

the sign of which is given by the sign of $\eta_g(x) - \eta_{g^*}(x)$. Thus the second derivative is positive (negative) if η_g is greater (less) than η_{g^*} for all x. This proves Theorem 1.

The following corollaries are immediately derived from Theorem 1:

COROLLARY 1: *The concentration curve for the function $g(x)$ will be above (below) the egalitarian line if $\eta_g(x)$ is less (greater) than zero for all $x \geqslant 0$.*

COROLLARY 2: *The concentration curve for the function $g(x)$ lies above (below) the Lorenz curve for the distribution of x if $\eta_g(x)$ is less (greater) than unity for all $x \geqslant 0$.*

The proof of Corollary 1 is also given by Roy, Chakravarty, and Laha [**12**].

It should be pointed out that the concentration curve for $g(x)$ is not the same thing as the Lorenz curve for $g(x)$. Both are identical if $g(x)$ is strictly monotonic and has a continuous derivative $g'(x)>0$ for all x (see Kakwani [**5**] for the proof).

The concentration index for $g(x)$ is defined as one minus twice the area under the concentration curve for $g(x)$. In our notion, it is the following:

$$C_g = 1-2\int_0^{\infty} F_1[g(x)]f(x)\,dx. \tag{2.4}$$

It is to be noted that if $g(x)=$ constant, the concentration curve coincides with the egalitarian line so that $C_g=0$. Further, if $g'(x)>0$ for all x, then C_g is always positive and will be equal to the Gini index of the function $g(x)$. Finally, if $g'(x)<0$ for all x, then the concentration curve for $g(x)$ is above the egalitarian line and C_g will be equal to minus one times the Gini index of $g(x)$. If $g(x)$ is not a monotonic function, then C_g lies between $-G_g$ and $+G_g$, where G_g is the Gini index of the function $g(x)$.

THEOREM 2: *If $g(x)=\sum_{i=1}^k g_i(x)$ so that $E[g(x)]=\sum_{i=1}^k E[g_i(x)]$ where E is the expected value operator, then*

$$E[g(x)]F_1[g(x)] = \sum_{i=1}^{k} E[g_i(x)]F_1[g_i(x)] \tag{2.5}$$

PROOF OF THEOREM 2: Substituting $g(x)=\sum_{i=1}^k g_i(x)$ in (2.1) gives

$$F_1[g(x)] = \frac{1}{E[g(x)]}\sum_{i=1}^{k}\int_0^x g_i(x)f(x)\,dx. \tag{2.6}$$

Now $F_1[g_i(x)]$ is given by

$$F_1[g_i(x)] = \frac{1}{E[g_i(x)]}\int_0^x g_i(x)f(x)\,dx, \tag{2.7}$$

which on substituting in (2.6) gives the result stated in Theorem 2.

Now using (2.4) in (2.5) we arrive at the following corollary.

COROLLARY 3: *If $g(x)=\sum_{i=1}^k g_i(x)$ and C_g and C_{gi} are concentration indices of $g(x)$ and $g_i(x)$, respectively, then*

$$E[g(x)]C_g = \sum_{i=1}^{k} E[g_i(x)]C_{gi}. \tag{2.8}$$

It should be noted that the results stated in Theorem 2 and Corollary 3 do not require the functions $g(x)$ and $g_i(x)$ to be monotonic. However, if we assume that $g(x)$ is a nondecreasing function of x, then $C_g = G_g$. Further, if $g_i(x)$ is any function of x (not necessarily monotonic), then $C_{gi} \leqslant G_{gi}$, where G_{gi} is the Gini index of $g_i(x)$. Thus from Corollary 3 we have

$$(2.9) \qquad E[g(x)]G_g \leqslant \sum_{i=1}^{k} E[g_i(x)]G_{gi}.$$

Note that if all $g_i(x)$ are nondecreasing functions of x, then the Gini index of $g(x)$ will be exactly equal to the weighted average of the Gini indices of individual $g_i(x)$.

3. SOME APPLICATIONS OF THE THEOREMS

In this section we shall consider some of the application of the theorems given in the last section.[3]

3.1. *The Engel Curve*

If $g(x)$ is the equation of Engel curve of a commodity, then it follows from Corollaries 1 and 2 that if its concentration curve lies above the egalitarian line, it is an inferior commodity; if the concentration curve lies between the Lorenz curve of x and the egalitarian line, it is a necessary commodity; and if the concentration curve lies below the Lorenz curve, the commodity is a luxury.

3.2. *Consumption and Saving Functions*

In the Keynesian case the consumption is related to income either linearly or curvilinearly. If the average propensity to consume decreases as income rises, it implies that the income elasticity of consumption will be less than one and the income elasticity of saving will be greater than one. If both consumption and saving are nondecreasing functions of income, then it follows from Corollary 2 that the inequality of income is greater than that of spending and less than that of saving.

3.3. *Effect of Direct Taxes on the Income Distribution*

Let x be the pre-tax income of an individual and $T_1(x)$ the tax function, then the disposable income is given by:

$$(3.3.1) \qquad d_1(x) = x - T_1(x),$$

[3] Many more applications of the theorems will be discussed in a forthcoming monograph which is in preparation.

which is an increasing function of x if the marginal tax rate is less than one. Using Theorem 2 in (3.3.1) and simplifying the result we obtain

$$(3.3.2)\quad F_1[d_1(x)]-F_1(x)=\frac{e_1}{(1-e_1)}[F_1(x)-F_1(T_1(x)],$$

where e_1 is the average tax rate of the society.

If the tax function $T_1(x)$ is such that the average tax rate increases as income increases, then the tax elasticity will be greater than one for all x which from Corollary 2 implies that $F_1(x)>F_1[T_1(x)]$ for all x. This leads to the conclusion from (3.3.2) that after-tax income will be more equally distributed than before-tax income.[4] The interesting point to note here is the marginal tax rate need not increase for this result to hold. Needless to say, the increasing marginal tax rate is a stronger condition than the increasing average tax rate.

The distance between $F_1(x)$ and $F_1[T_1(x)]$ depends on the tax elasticity. It follows from Theorem 1 that the larger the tax elasticity, the greater is the distance between $F_1(x)$ and $F_1[T_1(x)]$. Therefore, the equation (3.3.2) implies that the distance between the Lorenz curve of pre-tax and post-tax incomes depends on the two factors, viz, the tax elasticity and the average tax rate. It can be seen that the larger the average tax rate with a given tax elasticity or progressivity, the more equal will be the post-tax income distribution. Since the average tax rate can be changed without changing the tax elasticity or the progressivity, it follows that by simply comparing the Lorenz curves of pre-tax and post-tax incomes one cannot arrive at a suitable measure of progression.

Let $T_2(x)$ be any other tax function which gives the disposable income $d_2(x)$ and the average tax rate e_2; then from (3.3.2) we can write

$$(3.3.3)\quad F_1[d_1(x)]-F_2[d_2(x)] =\frac{(e_1-e_2)F_1(x)}{(1-e_1)(1-e_2)}+\frac{e_2F_2[T_2(x)]}{(1-e_2)}-\frac{e_1F_1[T_1(x)]}{(1-e_1)},$$

which implies from Theorem 1 that if there are two tax functions yielding the same average tax rate, the tax function with uniformly higher tax elasticity will give the post-tax income distribution more equal than the tax function with lower tax elasticity. Similarly, if the tax functions have the same tax elasticity or progressivity, the tax function with higher average tax rate gives the post income distribution more equal than the tax function giving lower average tax rate.

3.4. *Income Inequality by Factor Components*

Suppose the total family income x is written as the sum of n factor incomes $x_1, x_2, \ldots, x_n$. Let us denote $g_i(x)$ to be equal to the mean ith factor income of the

[4] One of the referees drew my attention to an unpublished paper by Jakobsson [4] who has also proved this result.

families having the same total income x. Then we have

$$(3.4.1) \quad x = \sum_{i=1}^{n} g_i(x).$$

If the families are arranged according to their total income x, then $F_1[g_i(x)]$ is interpreted as the proportion of the ith factor income of the families having total income less than or equal to x. Applying Theorem 2 to (3.4.1) we obtain the Gini index of the total family income equal to

$$(3.4.2) \quad G = \frac{1}{\mu} \sum_{i=1}^{n} \mu_i C_{gi},$$

where C_{gi} is the concentration index of the mean ith factor income $g_i(x)$ and μ_i is the mean of the ith factor income of all the families.

The factor income x_i is not necessarily an increasing function of x; therefore, it follows from equation (2.9) that

$$(3.4.3) \quad G \leqslant \frac{1}{\mu} \sum_{i=1}^{n} \mu_i G_i$$

where G_i is the Gini index of the ith factor income. Thus the weighted average of the factor income Gini indices provides an upper bound of the Gini index of the total income.[5]

Equation (3.4.2) can be used to analyze the contribution of each factor income to the total income inequality. To illustrate this numerically we utilize the data obtained from the Australian Survey of Consumer Expenditure and Finance, 1967–1968 (see Podder and Kakwani [**10**] for the basic data). The data were available in grouped form. The concentration index for each factor income was computed by fitting the equation of the Lorenz curve recently proposed by

TABLE I

INEQUALITY BY FACTOR INCOME COMPONENTS

Factor income	Mean income	Concentration index	Contribution of each factor of income inequality to the total income inequality	Per cent of cash contribution
Employment	3399	.3449	.3014	82.68
Unicorp.	276	.5852	.0415	11.38
Property	106	.4322	.0118	3.24
Regular Annuity	37	.0544	.00052	.14
Capital items	18	.3737	.00169	.46
Capital gains	39	.7737	.007776	2.13
Miscellaneous	14	.2922	.001037	.28
Total income	3890	.3645	.3645	100

[5] The result given in (3.4.3) has been derived earlier by Rao [**11**]. The referee drew my attention to this paper.

Kakwani and Podder [6]. The results are presented in Table I.[6] It is seen from Table I that income from employment, i.e., wages and salaries, contributes 82.68 per cent to the total income inequality. Unincorporated business income is second, contributing 11.38 per cent and the property income, i.e., interest, dividends, and rent contribute only 3.24 per cent of the total inequality.

3.5. *Income Inequality and Prices*

We now consider the effect of price changes on the income inequality of real income.

The demand equations of the linear expenditure system are derived by maximizing the Klein and Rubin [7] form of the utility function. The indirect utility function of this system is obtained as in Goldberger [3]:

$$(3.5.1)\quad u=\sum_{i=1}^{n}\beta_i\log\beta_i+\log(v-a)-\sum_{i=1}^{n}\beta_i\log p_i$$

where v is the total expenditure, p_i is the price of the ith commodity, $a=\sum_{i=1}^{n}p_i\gamma_i$ is interpreted as the subsistance total expenditure, and β_i are marginal budget shares.

Suppose the prices p_i change to p_i^*, and total expenditure v changes to v^*; then the resulting change in the utility will be

$$(3.5.2)\quad \Delta u=\log(v^*-a^*)-\log(v-a)-\sum_{i=1}^{n}\beta_i(\log p_i^*-\log p_i),$$

where $a^*=\sum_{i=1}^{n}p_i^*\gamma_i$. If the change in utility is set equal to zero, we obtain the total per capita expenditure v^* so that the family maintains the same level of utility:

$$(3.5.3)\quad v^*=a^*+(v-a)\prod_{i=1}^{n}\left(\frac{p_i^*}{p_i}\right)^{\beta_i},$$

v^* will be the real expenditure. Let G^* be the Gini index of the real expenditure; then Theorem 2 gives

$$(3.5.4)\quad G^*=\frac{\prod_{i=1}^{n}\left(\frac{p_i^*}{p_i}\right)^{\beta_i}\mu G}{a^*+(\mu-a)\prod_{i=1}^{n}\left(\frac{p_i^*}{p_i}\right)^{\beta_i}},$$

where μ and G are the mean and the Gini index of the money expenditure in the base year.

[6] It should be noted that only the concentration indices can be computed for each factor income from the grouped data because the families are ranked only according to the total income. The computations of Gini indices require the ranking of families according to each factor income, which is generally not available. Fortunately, we require only the concentration indices. The Gini indices of factor incomes provide only the upper bound as shown in (3.4.3).

TABLE II

INDEX OF INCOMES INEQUALITY IN THE UNITED KINGDOM: 1964–1972

Year	True cost of living index	Index of income inequality
1964	1.000	1.000
1965	1.046	.9952
1966	1.085	.9898
1967	1.112	.9892
1968	1.164	.9931
1969	1.223	.9894
1970	1.290	.9907
1971	1.391	.9885
1972	1.469	.9823

It is obvious from (3.5.4) that if all prices change in the same proportion, the income inequality is unaffected. The ratio v^*/v is the true cost of living index; it converts the money expenditure into real expenditure. In the spirit of the true cost of living index, we propose to use the ratio G^*/G as an index of income inequality to take into account the effect of relative price changes. This index converts the inequality of the money household expenditure distribution to the inequality of the real household expenditure. If this index is less than one, it implies that the relative price changes are making the expenditure distribution more unequal.

The numerical results on the index of income inequality are presented in Table II. The United Kingdom data were used for this purpose (Muellbauer [**9**]). It is seen from the table that the relative price changes from 1964 to 1972 have the effect of increasing income inequality; the 1971–1972 change is particularly marked.

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