Optimal Biodiverse Pricing Model

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(1.) $$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) = rx - \frac{rx^2}{K} = f(x)$$, where:

r = the intrinsic rate of growth
x = the stock of the renewable resource
K = the asymptotic limit, or carrying capacity, of the resource

(2.) $$\frac{d^2x}{dt} \rightarrow x = \frac{K}{2}$$, that is, the maximum sustainable yield will be where the stock of the renewable resource is at one-half of the carrying-capacity level.

(3.) $$\frac{dx}{x(K-x)} = \frac{r}{K}dt, \text{ or} \left(\frac{1}{x} + \frac{1}{K-x}\right)dx = rdt$$

Integrating (3.) yields:

(4.) $$\ln\left(\frac{x}{K-x}\right) = rt + \ln\left(\frac{x_0}{K-x_0}\right),$$ where $x_o = x(0)$.

Equation (4.) can now be written as:

(5.) $$x(t) = \frac{K}{1 + ce^{-rt}}$$, where $c = \frac{K - x_0}{x_0}$, the basic logistic equation for the renewable resource.

(6.) $$NPV = \int_0^{\infty} e^{-\delta t} R(x,E)dt = \int_0^{\infty} e^{-\delta t}\{p - c[x(t)]\}h(t)dt$$ subject to x(t)≥0 and h(t)≥0.

Since this integral has the form $\int \varphi(t, x, \dot{x})dt$, the necessary condition for a maximum will be:

(7.) $$\frac{\partial\phi}{\partial x} = \frac{d}{dt}\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left\{e^{-\partial t}[p - c(x)][F(x) - \dot{x}]\right\}$$

(8.) $$= e^{-\partial t}\left\{-c'(x)[F(x) - \dot{x}] + [p - c(x)F(x)]\right\}$$

(9.) $$\frac{\partial}{\partial t}\frac{\partial\phi}{\partial x} = \frac{d}{dt}\left\{e^{-\partial t}[p - c(x)]\right\} = e^{-\partial t}\left\{\delta[p - c(x)] + c'(x)\dot{x}\right\}$$

(10.) $$-c'(x)F(x) + [p - c(x)]F'(x) = \delta[p - c(x)] \rightarrow F'(x) - \frac{c'(x)F(x)}{p - c(x)} = \delta.$$

(11.) $L = \sum_{t=o}^{n} \rho^t \left\{ F(X) + \rho\lambda_{t+1}\left[R_t - q_t - R_{t+1}\right]\right\}$, where:

F(X) = the harvesting rate of the resource, depends on the intrinsic rate of growth and the harvest rate, where:

ρ^t = the discount factor at time t, and
λ_0 = the shadow price of the resource.

(12.) $X^* = \frac{K(r-\delta)}{2r}$. From this expression, with Y*=rX*(1-X*/K), we derive the optimal harvest rate as:

(13.) $Y^* = \frac{K\left(r^2 - \delta^2\right)}{4r}$. With $\rho\lambda = \pi'(Y)$, we get $\lambda = (1+\delta)(a - bY^*)$, or

(14). $\lambda^* = (1+\delta)\left[a - bK\left(r^2 - \delta^2\right)/(4r)\right]$.

(15.) $\frac{dx}{dt} = rx\left(1 - \frac{X}{K}\right) - q_1 Ex$

(16.) $\frac{dy}{dt} = sy\left(1 - \frac{Y}{L}\right) - q_2 Ey$, where:

r, s = intrinsic rates of growth of species X and Y, respectively,
K,L = the environmental carrying capacity of species X and Y, respectively,
q1, q2 = the catchability coefficients of the two populations, and
E = the harvesting effort for each species

We can further stipulate that the basic market prices of each species are constant at p1 and p2, in which case, the net benefit function can be defined as:

(17.) $\pi(x, y, E) = p_1 q_1 xE + p_2 q_2 yE - cE$

In equilibrium dx/dt and dy/dt must be equal to zero. As long as $0 \le x \le K$ and $0 \le y \le L$, then the bionomic equilibrium will be defined by:

(18.) $\frac{r}{q_1}\left(1 - \frac{x}{K}\right) = \frac{s}{q_2}\left(1 - \frac{y}{L}\right)$. If $r/q_1 < s/q_2$, then the equilibrium will be given as

(19.) $\tilde{y} = \left(1 - \frac{rq_2}{sq_1}\right)$, where $\tilde{y}$ is the minimum stock of y consistent with a positive stock of x.

(20.) $PV = \int_0^\infty d^{-\delta t}\left[p_1 q_1 x + p_2 q_2 y - c\right]E(t)dt$, subject to the constraints defined above and, the control constraint $0 \le E(t) \le E_{max}$. The corresponding Hamiltonian thus is:

$$\begin{aligned} H &= e^{-\partial t}\left[p_1 q_1 x + p_2 q_2 y - c\right]E + \lambda_1(t)\left[F(x) - q_1 E x\right] + \lambda_2(t)\left[G(y) - q_2 E y\right] \\ &= \sigma(t)E + \lambda_1 F(x) + \lambda_2 G(y) \end{aligned} \tag{21.}$$

where the respective lambda expressions are adjoint variables. The corresponding equations are:

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -e^{-\partial t} p_1 q_1 E - \lambda_1\left[F'(x) - q_1 E\right], \text{ and} \tag{22.}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = -e^{-\partial t} p_2 q_2 E - \lambda_2\left[G'(y) - q_2 E\right]. \tag{23.}$$

Equilibrium requires that:

$$E = \frac{F(x)}{q_1 x} = \frac{G(y)}{q_2 y}. \tag{24.}$$

Under this condition, equations (24.) and (25.) reduce to:

$$\frac{d\lambda_1}{dt} - \gamma_1 \lambda_1 = -p_1 q_1 E e^{-\partial t} \tag{25.}$$

$$\frac{d\lambda_2}{dt} - \gamma_2 \lambda_2 = -p_2 q_2 E e^{-\partial t}. \tag{26.}$$

Since $-\gamma_1 = F'(x) - \frac{F(x)}{x} = \frac{rx}{K}$ and $-\gamma_2 = \frac{sy}{L}$, the solutions are:

$$e^{\partial t}\lambda_1(t) = \frac{p_1 q_1 E}{\gamma_1 + \delta} = \text{constant}, \text{ and} \tag{27.}$$

$$e^{\partial t}\lambda_2(t) = \frac{p_2 q_2 E}{\gamma_2 + \delta} = \text{constant}. \tag{28.}$$

As long as shadow prices are bounded at t approaches infinity, and as long as optimal equilibrium satisfies $0 \le E \le E_{max}$, then a closed form solution can be derived such that:

$$\frac{\partial H}{\partial E} = e^{-\partial t}\left(p_1 q_1 x + p_2 q_2 y - c\right) - \lambda_1 q_1 x - \lambda_2 q_2 y = 0. \tag{29.}$$

From equation (26.), equations (28.) and (29.),

$$p_1 q_1 \left[x - \frac{F(x)}{\gamma_1 + \delta}\right] + p_2 q_2 \left[y - \frac{G(y)}{\gamma_2 + \delta}\right] = c \tag{30.}$$