A NOTE ON THE VALUES OF INDEPENDENCE POLYNOMIALS AT -1

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ABSTRACT. The *independence polynomial* $I(G;x)$ of a graph G is $I(G;x)=\sum_{k=0}^{\alpha(G)} s_k x^k$, where s_k is the number of independent sets in G of size k. The *decycling number* of a graph G, denoted $\phi(G)$, is the minimum size of a set $S \subseteq V(G)$ such that $G-S$ is acyclic. Engström proved that the independence polynomial satisfies $|I(G;-1)| \leq 2^{\phi(G)}$ for any graph G, and this bound is best possible. Levit and Mandrescu provided an elementary proof of the bound, and in addition conjectured that for every positive integer k and integer q with $|q| \leq 2^k$, there is a connected graph G with $\phi(G)=k$ and $I(G;-1)=q$. In this note, we prove this conjecture.

1. Introduction

Let $\alpha(G)$ denote the *independence number of a graph* G, the maximum order of an independent set of vertices in G. The *independence polynomial of a graph* G is given by

$$I(G;x)=\sum_{k=0}^{\alpha(G)} s_k x^k,$$

where s_k is the number of independent sets of size k in G. The independence polynomial has been the object of much research (see for instance the survey [7]). One direction of this research, partly motivated by connections with hard-particle models in physics [1, 2, 3, 5, 6], has focused on the evaluation of the independence polynomial at $x=-1$.

The *decycling number of a graph* G, denoted $\phi(G)$, is the minimum size of a set of vertices $S \subseteq V(G)$ such that $G-S$ is acyclic. Engström [3] proved the following bound on $I(G;-1)$, which is best possible.

Theorem 1.1 (Engström)**.** *For any graph G, $|I(G;-1)| \leq 2^{\phi(G)}$.*

Levit and Mandrescu [8] gave an elementary proof of Theorem 1.1 and, in addition, proposed the following conjecture.

Conjecture 1 (Levit and Mandrescu)**.** Given a positive integer k and an integer q with $|q| \leq 2^k$, there is a connected graph G with $\phi(G)=k$ and $I(G;-1)=q$.

For brevity, in this paper a graph G with $\phi(G)=k$ and $I(G;-1)=q$, with $|q| \leq 2^k$, will be referred to as a (k,q)*-graph*. In [9], Levit and Mandrescu provided constructions that gave (k,q)-graphs for all $k \leq 3$ and $|q| \leq 2^k$. Also, they gave constructions for every k provided $q \in \{2^{\phi(G)}, 2^{\phi(G)}-1\}$. In this paper, we prove Conjecture 1.

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2. The Construction and Proof of Conjecture

The construction proceeds inductively, using particular $(k-1,q)$-graphs to produce the necessary (k,q)-graphs. First we assemble the tools used in the construction. The most important tool is a recursive formula for $I(G;x)$ due to Gutman and Harary [4]. We let $N(v) = \{x \in V(G) : xv \in E(G)\}$ and $N[v] = \{v\} \cup N(v)$.

Lemma 2.1. *For any graph G and any vertex $v \in V(G)$,*

$$I(G;x) = I(G-v;x) + xI(G-N[v];x).$$

Using this, or simply counting independent sets, we can derive the independence polynomial at -1 for small graphs. Some useful examples can be found in Table 2.

G	$I(G;-1)$
K_1	0
K_2	-1
$K_3 = C_3$	-2
C_6	2

Table 1. Some small examples

Since Lemma 2.1 requires a particular vertex $v \in V(G)$ to be specified, it will often be helpful to root graphs for which we want to compute the independence polynomial at -1. Given a graph G and a vertex $v \in V(G)$, the *rooted graph* G_v is the graph G with the vertex v labeled. Of course, $I(G;-1) = I(G_v;-1)$ for any vertex $v \in V(G)$.

We now introduce two operations on rooted graphs which will be useful in our proof. The first of these is called *pasting*.

Definition. Given two rooted graphs G_v and H_w, the *pasting of G_v and H_w*, denoted $G_v \wedge H_w$, is the rooted graph formed by identifying the roots v and w.

We note two important facts. First, the pasting operation creates no new cycles, and thus $\phi(G_v \wedge H_w) \le \phi(G_v) + \phi(H_w)$. (In our construction the roots will be pendant vertices, and so $\phi(G_v \wedge H_w) = \phi(G_v) + \phi(H_w)$.) Second, if for two rooted graphs G_v and H_w the quantities $I(G_v;-1)$ and $I(H_w;-1)$ have been evaluated using Lemma 2.1, then the value of $I(G_v \wedge H_w;-1)$ can be determined in a straightforward way. It is well-known that, letting $G \cup H$ denote the disjoint union of G and H, we have

$$I(G \cup H;x) = I(G;x)I(H;x).$$

Deleting the pasted vertex in $G_v \wedge H_w$ produces a disjoint union of graphs. This fact, and the recurrences

$$I(G_v;-1) = I(G_v - v;-1) - I(G_v - N[v];-1)$$
$$I(H_w;-1) = I(H_w - w;-1) - I(H_w - N[w];-1)$$

then give

$$I(G_v \wedge H_w;-1) = I(G_v - v;-1)I(H_w - w;-1) - I(G_v - N[v];-1)I(H_w - N[w];-1).$$

It will be helpful to keep track of the various parts of the above calculation, and in order to do so we introduce the following bookkeeping device. Given a rooted graph G_v, where $I(G_v - v;-1) = a$ and $I(G_v - N[v];-1) = b$, and hence $I(G_v;-1) = a - b$, we write $I(G_v;-1) = \langle a-b, a, b\rangle$ and say that G_v has *bracket* $\langle a-b, a, b\rangle$. An example can be found in Figure 1. Note that for a given rooted

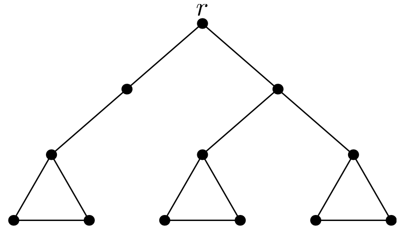

FIGURE 1. A graph rooted at r with bracket $\langle 5, -3, -8 \rangle$.

graph G_v there are unique integers a and b, determined by the root, with $I(G_v; -1) = \langle a-b, a, b\rangle$. Using this notation, the calculations above give the following lemma.

Lemma 2.2 (Pasting Lemma)**.** *If G_v and H_w are rooted graphs on at least two vertices with $I(G_v; -1) = \langle a-b, a, b\rangle$ and $I(H_w; -1) = \langle c-d, c, d\rangle$, then*

$$I(G_v \wedge H_w; -1) = ac - bd = \langle ac - bd, ac, bd\rangle$$

and $G_v \wedge H_w$ has bracket $\langle ac - bd, ac, bd\rangle$.

Our second operation is a variation of the pasting operation which, however, is useful enough to merit its own terminology and notation.

Definition. Given a rooted graph G_v and an integer $k \geq 0$, the *ℓ-extension of G_v*, denoted G_v^ℓ is the graph formed by identifying the root v with one of the endpoints of a (disjoint) path of length ℓ and reassigning the root to the other endpoint of the path.

The length of a path is above measured in edges; for instance for a rooted graph G_v, the 0-extension G_v^0 is simply G_v. As with the pasting operation, no new cycles are created by the extension operation, and so here $\phi(G_v^\ell) = \phi(G)$ for any ℓ. In addition, the values of the independence polynomial at -1 of various extensions of a rooted graph G_v are easy to characterize in terms of the bracket of G_v. Indeed, extensions of G_v have the same bracket values, up to sign, but in a different order. The proof of the following lemma follows immediately from the recursion formula and is omitted.

Lemma 2.3 (Extension Lemma)**.** *If G_v is a rooted graph with $I(G_v; -1) = \langle a-b, a, b\rangle$, then*

$$I(G_v^1; -1) = \langle -b, a-b, a\rangle$$
$$I(G_v^2; -1) = \langle -a, -b, a-b\rangle$$

and $I(G_v^3; -1) = \langle b-a, -a-b\rangle = -\langle a-b, a, b\rangle = -I(G_v; -1)$.

We illustrate the cycling phenomenon with C_6, a graph which will be used in our construction, in Table 2. Obviously we may consider C_6 rooted at any given vertex.

(Since C_3 has the same set of six brackets, in a different order, when extended, C_3 could also have been used in the constructions and proofs to come. We choose C_6 solely because C_6^0 and C_6^1 have positive $I(G; -1)$.)

Using the pasting and extension operations we have our final lemma, which shows that the word "connected" in the conjecture is superfluous. Any disconected (k, q)-graph can be pasted together and extended to produce a connected (k, q)-graph.

Lemma 2.4. *Let G and H be disjoint (k_1, q_1) and (k_2, q_2)-graphs, respectively, with $k_1 + k_2 = k$ and $q_1 q_2 = q$. Then there is a connected (k, q)-graph F, i.e., F is connected, $\phi(F) = k_1 + k_2 = k$, and $I(F; -1) = q_1 q_2 = I(G \cup H; -1)$.*

ℓ	$I(C_6^\ell;-1)$
0	$\langle 2,1,-1\rangle$
1	$\langle 1,2,1\rangle$
2	$\langle -1,1,2\rangle$
3	$\langle -2,-1,1\rangle$
4	$\langle -1,-2,-1\rangle$
5	$\langle 1,-1,-2\rangle$
6	$\langle 2,1,-1\rangle$

TABLE 2. Brackets of C_6^ℓ

Proof. Root the given graphs as G_v and H_w and let the corresponding brackets be $I(G_v;-1) = \langle q_1,a,b\rangle$ and $I(H_w;-1) = \langle q_2,c,d\rangle$, respectively. Let $F' = (G_v^2 \wedge H_w^2)^1$. By the Extension Lemma, $I(G_v^2;-1) = \langle -a,-b,q_1\rangle$ and $I(H_w^2;-1) = \langle -c,-d,q_2\rangle$. Then, by the Pasting Lemma,

$$I(G_v^2 \wedge H_w^2;-1) = \langle bd - q_1q_2, bd, q_1q_2\rangle$$

Therefore, again using the Extension Lemma,

$$\begin{aligned} I(F';-1) &= I((G_v^2 \wedge H_w^2)^1;-1) \\ &= \langle -q_1q_2, bd - q_1q_2, bd\rangle \\ &= -q_1q_2 \\ &= -I(G \cup H;-1). \end{aligned}$$

Thus, if we let $F = (G_v^2 \wedge H_w^2)^4$, we see that

$$I(F;-1) = \langle q_1q_2, q_1q_2 - bd, bd\rangle = q_1q_2 = I(G \cup H;-1).$$

In addition, neither the pasting nor extension operations produce cycles, so $\phi(F) = k_1 + k_2 = k = \phi(G \cup H)$. ☐

By setting $H = K_2$ and $H = C_6$ in Lemma 2.4 in turn, we obtain the following facts, which will also be useful in the proof. These two facts were also noted by Levit and Mandrescu [9], who used different *ad hoc* techniques in their constructions of the necessary graphs.

Corollary 2.5. *If G is a (k,q)-graph then there exists (a) a connected $(k+1,2q)$-graph and (b) a connected $(k,-q)$-graph.*

We now prove Conjecture 1.

Theorem 2.6. *Given a positive integer k and an integer q with $|q| \leq 2^k$, there is a connected graph G with $\phi(G) = k$ and $I(G;-1) = q$.*

Proof. By Lemma 2.4 we do not need to produce connected (k,q)-graphs for all $|q| \leq 2^k$; disconnected (k,q)-graphs will suffice. Since $I(G \cup K_1;-1) = 0$ for all G, we can consider the case $q = 0$ done for all k.

As mentioned previously, our proof proceeds inductively on k. When $k = 1$ then $I(C_6;-1) = \langle 2,1,-1\rangle$ and, as noted in Table 2, by taking extensions of C_6, we rotate through all of $\{2,1,-1,-2\}$. Thus the theorem is true for $k = 1$.

For the induction step, assume $(k-1,q)$-graphs are constructible for all q with $|q| \leq 2^{k-1}$. By Corollary 2.5(a) we immediately have that (k,q)-graphs for even q with $|q| \leq 2^k$ are constructible. By Corollary 2.5(b) we also need only construct (k,q)-graphs for positive $q \leq 2^k$. It only remains, then, to construct (k,q)-graphs for q each odd integer in $[0,2^k]$. To that end, we prove the following claim.

Claim 1. For each odd integer $q \in [0,2^k]$, there is a connected (k,q)-graph G_v such that either $I(G_v;-1) = \left\langle q, 2^k, 2^k - q \right\rangle$ or $I(G_v;-1) = \left\langle q, -2^k + q, -2^k \right\rangle$.

Proof. For $k = 1$, we see that the bracket of C_6^1 has the necessary form, i.e. $I(C_6^1;-1) = \langle 1,2,1 \rangle$. Assume that the hypothesis of the claim is true for $k-1$; we seek to produce (k,q)-graphs for each odd $q \in [0,2^k]$ such that 2^k or -2^k appears in their bracket. We consider two cases: $q \in [2^{k-1},2^k]$ and $q \in [0,2^{k-1}]$.

For the first case, let q be an odd integer in $[2^{k-1},2^k]$. Necessarily then, $q = 2^k - r$ for some $r \in [0,2^{k-1}]$. By the induction assumption, there is some $(k-1,r)$-graph G_v such that either $I(G_v;-1) = \left\langle 2^{k-1} - r, 2^{k-1}, r \right\rangle$ or $I(G_v;-1) = \left\langle 2^{k-1} - r, -r, -2^{k-1} \right\rangle$. By the Pasting Lemma, then, $I(G_v \wedge C_6^1;-1) = \left\langle 2^k - r, 2^k, r \right\rangle = q$ if the bracket of G_v is of the first form, or $I(G_v \wedge C_6^2;-1) = \left\langle 2^k - r, -r, -2^k \right\rangle$ if the bracket of G_v is of the second form. Thus the claim is true for all $q \in [2^{k-1},2^k]$.

For the second case, when q is an odd integer in $[0,2^{k-1}]$, note that $q+q' = 2^k$ for some odd integer $q' \in [2^{k-1},2^k]$. Thus we can simply apply the Extension Lemma (as many times as necessary) to the examples produced for the first case. $\square$

The proof of the claim completes the induction, and completes the proof. $\square$

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