THE INTERLACE POLYNOMIAL OF GRAPHS AT -1

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Abstract. In this paper we give an explicit formula for the interlace polynomial at $x = -1$ for any graph, and as a result prove a conjecture of Arratia, Bollobás and Sorkin [2] that states that it is always of the form $\pm 2^s$. We also give a description of the graphs for which s is maximal.

1. Introduction

The study of Euler circuits of directed graphs related to DNA sequencing [1] inspired Arratia, Bollobás and Sorkin [2] to introduce a new graph polynomial satisfying a striking recurrence relation. Although in [3] a fair amount is proved about the interlace polynomial, it is still a rather mysterious graph invariant. The aim of this note is to shed more light on the interlace polynomial by proving a conjecture of Arratia, Bollobás and Sorkin [2]. Before we state this conjecture, we introduce the interlace polynomial.

As usual, we write $N_G(v)$ for the closed neighborhood of a vertex v in a graph G, i.e., $N_G(v) = \{u : uv \in E(G) \text{ or } u = v\}$. For any non-empty graph G and any edge $xy \in E(G)$ define $G^{(xy)}$, the *pivot* of G with respect to xy, as a graph with $V(G^{(xy)}) = V(G)$ and $E(G^{(xy)})$ equal to the symmetric difference $E(G) \triangle S$ where S is the set of edges uv with $u, v \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}$ and $N_G(u) \cap \{x, y\} \neq N_G(v) \cap \{x, y\}$.

For an arbitrary graph G, the *interlace polynomial* $q_G(x)$ is defined by

$$q_G(x) = \begin{cases} x^n & \text{if } G = E_n \text{ is empty;} \\ q_{G-x}(x) + q_{G^{(xy)}-y}(x) & \text{if } xy \in E(G). \end{cases} \tag{1}$$

It is shown in [2] that this gives a well defined polynomial on all simple graphs. Note in particular, if $G = G_1 \cup G_2$ is the disjoint union of two graphs G_1 and G_2 then

$$q_G(x) = q_{G_1}(x) q_{G_2}(x). \tag{2}$$

Numerical evidence led Arratia, Bollobás and Sorkin [3] conjecture that the absolute value of $q_G(-1)$ is always a non-negative integer power of 2. In [3] it is shown that the conjecture holds for circle-graphs. In fact, in that case the conjecture follows from a theorem of Martin [6] (see Las Vergnas [4], [5]). Here we prove the full conjecture by a completely different method. We shall also describe the graphs G of order n for which $|q_G(-1)|$ is maximal.

2. The value at -1

Theorem 1. *Let A be the adjacency matrix of G, $n = |V(G)|$ and let r be the rank of the Laplacian matrix $I + A$ over the field of two elements. Then*

$$q_G(-1) = (-1)^r 2^{n-r} = (-1)^n (-2)^{n-r} \tag{3}$$

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Proof. Using the defining relation (1) it is enough to prove the result for the empty graph E_n and to prove that for other graphs the expression $(-1)^n(-2)^{n-r}$ satisfies the recurrence in (1). For empty graphs the result is immediate since $q_G(x) = x^n$ and the rank of $I + A = I$ is n. The recurrence can be written in the form

$$(-1)^n q_G(-1) + (-1)^{n-1} q_{G-x}(-1) + (-1)^{n-1} q_{G^{(xy)}-y}(-1) = 0. \tag{4}$$

Hence we need to show that $(-2)^{s_1} + (-2)^{s_2} + (-2)^{s_3} = 0$ where s_1, s_2 and s_3 are the nullities of $I + A$ for the three graphs

$$G \qquad G - x \qquad G^{(xy)} - y. \tag{5}$$

This in turn holds when these nullities are of the form s, s and $s + 1$ in some order for some s. Let $xy \in E(G)$ and order the vertices in the form $x, y, v_{01}, \ldots, v_{0k_0} v_{11}, \ldots, v_{3k_3}$ where v_{0i} are the vertices adjacent to neither x nor y, v_{1i} are adjacent to y only, v_{2i} are adjacent to x only and v_{3i} are adjacent to both x and y. The matrices $I + A$ for the graphs (5) are now of the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & A_{00} & A_{01} & A_{02} & A_{03} \\ 0 & 1 & A_{10} & A_{11} & A_{12} & A_{13} \\ 1 & 0 & A_{20} & A_{21} & A_{22} & A_{23} \\ 1 & 1 & A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & A_{00} & A_{01} & A_{02} & A_{03} \\ 1 & A_{10} & A_{11} & A_{12} & A_{13} \\ 0 & A_{20} & A_{21} & A_{22} & A_{23} \\ 1 & A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & A_{00} & A_{01} & A_{02} & A_{03} \\ 0 & A_{10} & A_{11} & 1{+}A_{12} & 1{+}A_{13} \\ 1 & A_{20} & 1{+}A_{21} & A_{22} & 1{+}A_{23} \\ 1 & A_{30} & 1{+}A_{31} & 1{+}A_{32} & A_{33} \end{pmatrix} \tag{6}$$

Where 1 refers to a block of 1's of appropriate size and $A_{ij} = A_{ji}^T$. Using elementary row and column operations we can reduce these matrices to

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{00} & A_{01} & A_{02} & A_{03} \\ 1 & 0 & A_{10} & 1{+}A_{11} & A_{12} & 1{+}A_{13} \\ 1 & 0 & A_{20} & A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{30} & 1{+}A_{31} & A_{32} & 1{+}A_{33} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & A_{00} & A_{01} & A_{02} & A_{03} \\ 0 & A_{10} & 1{+}A_{11} & A_{12} & 1{+}A_{13} \\ 0 & A_{20} & A_{21} & A_{22} & A_{23} \\ 0 & A_{30} & 1{+}A_{31} & A_{32} & 1{+}A_{33} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & A_{00} & A_{01} & A_{02} & A_{03} \\ 1 & A_{10} & 1{+}A_{11} & A_{12} & 1{+}A_{13} \\ 1 & A_{20} & A_{21} & A_{22} & A_{23} \\ 0 & A_{30} & 1{+}A_{31} & A_{32} & 1{+}A_{33} \end{pmatrix} \tag{7}$$

without affecting their rank. In the first matrix we have added the second row to the first, fourth and sixth (blocks of) rows. Then the second column was added to the first, fourth and sixth (blocks of) columns, so as to clear the non-diagonal entries in the second row. For the other two matrices we have added the first row to the third and fifth blocks of rows so as to make the first column of these matrices have the given forms. Then the first column was added to the third and fifth blocks of columns so as to make the first row have the given forms. Since removing the second row and column in the first matrix does not affect its nullity, the Theorem will now follow from the following simple algebraic lemma. □

Lemma 2. *If $M = M^T$ is symmetric, b is a row vector and all matrices are over the field of two elements then the nullities of the following matrices are of the form s, s and $s+1$ in some order.*

$$\begin{pmatrix} 0 & b \\ b^T & M \end{pmatrix} \qquad \begin{pmatrix} 1 & b \\ b^T & M \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$$

Proof. Assume first that there exists a vector v with $vM = b$. By adding v_i times row i to row 1 and v_i times column i to column 1 in the first two matrices we are reduced to the case $b = 0$. (If $vb^T = 1$ then the top left entry will be changed in both matrices, however this just swaps the formats of these matrices.) In the case $b = 0$ the nullities of the three matrices are clearly $s+1$, s and s respectively where s is the nullity of M. Now assume b is linearly independent of the rows of M. Since M is symmetric, b^T is linearly independent of the columns of M. Hence the rank of the matrix $\begin{pmatrix} b^T & M \end{pmatrix}$ is exactly one more than the rank of M, and the first row of either of the first two matrices is linearly independent of the others. Hence the ranks of the three matrices are $r+2$, $r+2$ and $r+1$ where r is the rank of M. The nullities are therefore of the form s, s, $s+1$. □

3. The maximal values of $q_G(-1)$

In this section we will determine the graphs G of order n for which $q_G(-1)$ is large. By Theorem 1 this is equivalent to determining the graphs for which the rank of the Laplacian $I + A$ is small.

Given a graph G, we write $C_0(G) = \mathbb{F}_2^{V(G)}$ for the vector space generated by $V(G)$ over the field of two elements. We shall often regard $C_0(G)$ as the set of subsets $V(G)$ with addition defined by the symmetric difference. Similarly, if S is any set of vertices of G, we let $C_0(S) = \mathbb{F}_2^S$ be the vector subspace of $C_0(G)$ spanned by S.

As above, define the closed neighborhood map $N_G \colon V(G) \to C_0(G)$ by $N_G(v) = \{x : xv \in E(G) \text{ or } x = v\}$. We can then uniquely extend N_G to a linear vector space map $L_G \colon C_0(G) \to C_0(G)$. In the standard basis of $C_0(G)$, L_G is given by the Laplacian $I + A$. On the other hand, if we regard elements of $C_0(G)$ as subsets of $V(G)$, then L_G maps $S \subseteq V(G)$ to $\triangle_{v \in S} N_G(v)$.

Define the *rank* of G, denoted $r(G)$, to be the dimension of $\operatorname{Im} L_G$, where Im denotes the image of a map. In other words, the rank of G is the rank of the matrix $I + A$ over $\mathbb{F}_2$. Two vertices x and y of G are called *twins* if for all $z \neq x, y$, $xz \in E(G)$ if and only if $yz \in E(G)$.

Note that in the induced subgraph of G on S, say, $v \in S$ has even degree if and only if $v \in L_G(S)$. Therefore, $S \setminus L_G(S)$ is of even size. In particular, if $L_G(S) = \emptyset$ then $|S|$ is even. Furthermore, if $L_G(\{v, w\}) = \emptyset$ then $N_G(v) = N_G(w)$ and so v and w are adjacent twins.

Lemma 3. *Let G be a graph on n vertices. If $L_G(S) = \emptyset$ and $v \in S$, then $r(G) = r(G - v)$.*

Proof. Deleting v from G corresponds to deleting the row and column corresponding to v from the matrix $I + A$. However, if $v \in S$ and $L_G(S) = \emptyset$ then the row corresponding to v is just the sum of the rows corresponding to the elements of $S \setminus \{v\}$. Hence, removing this row does not affect the rank of the matrix $I + A$. Similarly, removing the column corresponding to v does not affect the rank. Hence $r(G - v) = r(G)$. □

In particular, the presence of adjacent twins does not affect the rank, so we will mainly consider graphs without adjacent twins, and will show that any such graph with rank bounded by k is highly structured. In particular, will show that there exists a graph O_k such that all such graphs are induced subgraphs of O_k.

Define a graph G to be *closed* if for all sets S of vertices with $|S|$ odd, there exists a unique vertex v such that $L_G(S) = N_G(v)$. Taking a set S consisting of a single vertex, we see that a closed graph has no adjacent twins. We shall show that graphs with no adjacent twins and rank k are contained as induced subgraphs in closed graphs of rank k and give a bound on the number of vertices in these graphs.

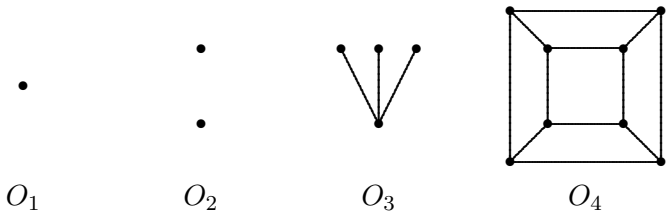

FIGURE 1. O_k for small k.

Lemma 4. *Let G be a graph on n vertices with no adjacent twins and of rank $r(G) = k$. Then $n \leq 2^{k-1}$, with equality if and only if G is closed.*

Proof. Let $v, w \in V(G)$. Then v and w are adjacent twins if and only if $N_G(v) = N_G(w)$. Therefore the map N_G is an injection and $|\operatorname{Im} N_G| = |V(G)| = n$. Let $\mathcal{E}$ be the subset of $C_0(G)$ consisting of sets S with $|S|$ even. Since the map $C_0(G) \to \mathbb{F}_2$ given by $S \mapsto |S| \bmod 2$ is linear, $\mathcal{E}$ is a linear subspace of $C_0(G)$ of codimension 1. The complement of $\mathcal{E}$ is $\mathcal{O}$, the set of all odd subsets of $V(G)$, and is a coset of $\mathcal{E}$. Since $L_G(S) \neq \emptyset$ for all $S \in \mathcal{O}$, the image $L_G(\mathcal{O})$ is a non-trivial coset of $L_G(\mathcal{E})$. Therefore $|L_G(\mathcal{O})| = |L_G(\mathcal{E})| = \frac{1}{2}|L_G(C_0(G))| = 2^{k-1}$. Since $\operatorname{Im} N_G$ is the image of L_G on one element sets, $\operatorname{Im} N_G \subseteq L_G(\mathcal{O})$ with equality if and only if G is closed. Hence $n \leq 2^{k-1}$ with equality if and only if G is closed. □

Lemma 5. *Suppose G is a graph with no adjacent twins. Then there exists a closed graph H such that G is an induced subgraph of H and $r(G) = r(H)$.*

Proof. Fix the rank k. By Lemma 4, the number of vertices of any counterexamples of this given rank is bounded. Consider a counterexample G of maximal order.

First note that G is clearly not closed. It cannot be the case that there exists S and distinct vertices v, w such that $L_G(S) = N_G(v) = N_G(w)$, since G has no adjacent twins. Therefore there exists $S \subseteq V(G)$ with $|S|$ odd such that there is no v with $L_G(S) = N_G(v)$.

Define a new graph G' as follows. Take a vertex w not in G, and connect it to the vertices in $L_G(S)$. We claim that G' has no adjacent twins and is of rank k.

Any pair of adjacent twins in G' must contain w since G had no such pair. Any twin v of w would have $N_G(v) = L_G(S)$, and by assumption, no such v exists. Hence there are no adjacent twins in G'.

Since $|S \setminus L_G(S)|$ is even and $|S|$ is odd, w is adjacent to an odd number of vertices in S. Hence $L_{G'}(S) = L_G(S) \cup \{w\} = L_{G'}(\{w\})$. Thus $L_{G'}(S \cup \{w\}) = \emptyset$ and so by Lemma 3, $r(G') = r(G' - w) = r(G) = k$.

Since G' satisfies the conditions above, and G is a maximal counterexample, there exists a closed graph H such that G' is an induced subgraph of H and $r(H) = k$. Therefore H also contains G as an induced subgraph, giving the desired contradiction. □

We now show that there is a unique closed graph of given rank, and we characterize its structure. Define the graph O_k to have vertex set $\{S \subseteq [k] : |S| \text{ is odd}\}$ and edge set given by $\{\{S, T\} : |S \cap T| \text{ is odd}\}$.

Theorem 6. *Let G be a closed graph with $r(G) = k$. Then G is isomorphic to O_k.*

Proof. We claim that G has an independent set of k vertices. Suppose not and let S be a independent set of vertices of maximal size. First, we will show the existence of a vertex $v \notin S$, adjacent to precisely one element w of S.

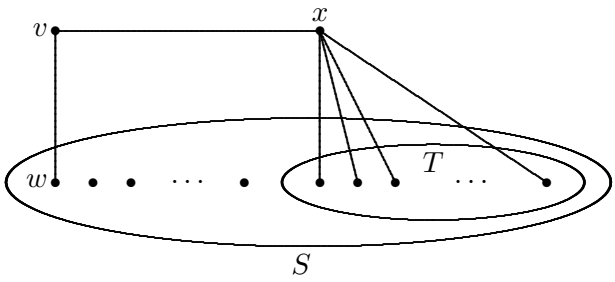

FIGURE 2. Structure of S'.

Define $N_S\colon V(G) \to C_0(S)$ by $N_S(x) = N_G(x) \cap S$. By maximality of S, there does not exist $x \in V(G)$ with $N_S(x) = \emptyset$, and so N_S is not surjective. Since $|S| < k$, we have by Lemma 4 that $|V(G)| = 2^{k-1} \geq |C_0(S)|$ and so N_S is not injective. Let x, y be such that $N_S(x) = N_S(y)$ and $x \neq y$. If $x \in S$, then $N_S(y) = \{x\}$ and we may set $v = y$ and $w = x$. (We know $y \notin S$ since y is adjacent to x.) If $x, y \notin S$, let $w \in S$. Since G is closed, there exists a vertex v such that $L_G(\{w, x, y\}) = N_G(v)$. Then $N_G(v) \cap S = N_S(w) = \{w\}$, so v, w have the required properties.

Note that the positions of v and w with respect to the set S are switched with respect to the maximal independent set $S \triangle \{v, w\}$ and that therefore v and w are symmetric.

As v and w are not adjacent twins, by symmetry we may assume that there is a vertex x adjacent to v and not w. Then x is not in S. Let $T = N_S(x)$ be the set of vertices in S adjacent to x, and let $S' = S \cup \{v, x\}$ (See Figure 2). Suppose that $|T|$ is even. Then by the closure property, there is a vertex z such that $N_G(z) = L_G(T \cup \{x\})$. Then $N_G(z) \cap S' = \{v, x\}$. Therefore z is not adjacent to any vertex in S, $z \notin S$ and $S \cup \{z\}$ is a larger independent set, causing a contradiction. If alternatively $|T|$ were odd, we would have z such that $N_G(z) = L_G(T \cup \{v, x\})$. Then $N_G(z) \cap S' = \{w, x\}$. Therefore $z \neq v$, $z \notin S$ and $S \cup \{z, v\} \setminus \{w\}$ is a larger independent set, causing a contradiction. These contradictions imply that there is an independent set S of size k.

Define $L_S\colon C_0(G) \to C_0(S)$ by $L_S(T) = L_G(T) \cap S$. For $i \in S$, $L_S(\{i\}) = N_S(i) = \{i\}$. Thus L_S acts as the identity on subsets of S. Hence $L_S(\mathcal{O}) = \operatorname{Im} N_S$ contains all 2^{k-1} odd subsets of S. However, $|\operatorname{Im} N_S| \leq |V(G)| = 2^{k-1}$, so N_S must induce a bijection between $V(G)$ and the set of all subsets of S of odd size.

Since S is a set of size k, we can identify the set of odd subsets of S with $V(O_k)$. The map N_S now induces a bijection from $V(G)$ to $V(O_k)$. We now prove that N_S gives an isomorphism between G and O_k.

Let v be any vertex of G. Since $N_S(v)$ is a subset of S of odd order and G is closed, there exists a vertex x such that $N_G(x) = L_G(N_S(v))$. Then $N_S(x) = L_S(N_S(v)) = N_S(v)$ and $v = x$. Therefore $N_G(v) = L_G(N_S(v))$, and for all vertices w, $w \in N_G(v)$ if and only if w is adjacent (or equal) to an odd number of elements of $N_S(v)$. In other words, w is adjacent or equal to v if and only if $|N_S(w) \cap N_S(v)|$ is odd. This is equivalent to $N_S(w)$ and $N_S(v)$ being adjacent or equal in O_k. □

Note that we have only shown that if there exists a closed graph of rank k, then this must be isomorphic to O_k. We have yet to show that O_k is a closed graph of rank k. We also note some other properties.

Corollary 7. *The graph O_k is a closed graph of rank k with the following properties:*

1. *O_k has 2^{k-1} vertices,*
2. *O_k is preserved by the pivot operation,*
3. *the automorphism group of O_k is transitive on the ordered independent sets of size k.*

Proof. If we apply Lemma 5 to the empty graph E_k, we see that there exists a closed graph of rank k, which by Theorem 6 must be O_k. The number of vertices is 2^{k-1} by Lemma 4. Since O_k is the unique graph of rank k with 2^{k-1} vertices without adjacent twins, all of which properties are preserved by the pivot operation, the graph itself is preserved by the pivot operation. Applying the proof of Theorem 6 to O_k gives an automorphism N_S of O_k taking any ordered independent set S of size k to $\{\{1\},\{2\},\ldots,\{k\}\}$. □

Finally, we get a definition of the rank of a graph with no adjacent twins in terms of the graphs O_k.

Corollary 8. *Let G be a graph with no adjacent twins. Then the rank of G is the smallest integer k such that G is an induced subgraph of O_k.*

Proof. We have shown in Lemma 5 and Theorem 6 that G is an induced subgraph of O_k. Assume $l < k$. Then G, which is of rank k cannot be an induced subgraph of O_l since, by Corollary 7, O_l is of rank l and induced subgraphs cannot have larger ranks than the whole graph. □

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