

EXTREMAL GRAPHS FOR HOMOMORPHISMS II

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ABSTRACT. Extremal problems for graph homomorphisms have recently become a topic of much research. Let $\text{hom}(G, H)$ denote the number of homomorphisms from G to H . A natural set of problems arises when we fix an image graph H and determine which graph(s) G on n vertices and m edges maximize $\text{hom}(G, H)$. We prove that if H is loop-threshold, then, for every n and m , there is a threshold graph G with n vertices and m edges which maximizes $\text{hom}(G, H)$. Similarly, we show that loop-quasi-threshold image graphs have quasi-threshold extremal graphs.

In the case $H = P_3^o$, the path on three vertices in which every vertex is looped, the authors [5] determined a set of five graphs, one of which must be extremal for $\text{hom}(G, P_3^o)$. Also in this paper, using similar techniques, we determine a set of extremal graphs for “the fox”, a graph formed by deleting the loop on one of the end-vertices of P_3^o . The fox is the unique connected loop-threshold image graph on at most three vertices for which the extremal problem was not previously solved.

1. INTRODUCTION

The study of extremal problems related to graph homomorphisms includes many areas of classical graph theory. For graphs G and H , we define $\text{Hom}(G, H)$ be the set of homomorphisms from G to H , i.e.,

$$\text{Hom}(G, H) = \{\phi : V(G) \rightarrow V(H) : xy \in E(G) \implies \phi(x)\phi(y) \in E(H)\}.$$

Also, we let $\text{hom}(G, H) = |\text{Hom}(G, H)|$. A natural set of problems arises from fixing an image graph H and determining which graph (or graphs) G from some class of graphs maximizes $\text{hom}(G, H)$ —see, for instance, [1, 5, 7, 8, 12, 15]. For example, setting $H = K_q$, we see that $\text{hom}(G, K_q)$ is simply the number of proper q -colorings of G . The problem of maximizing the number of q -colorings over graphs on n vertices and m edges was proposed by Linal [11] and Wilf [14]. This problem has been studied by many researchers—see the paper of Loh, Pikhurko, and Sudakov [12] for a detailed summary of research in this area.

Another image graph for which $\text{hom}(G, H)$ has been extensively studied is the path on two vertices with one vertex looped and the other unlooped, denoted H_I (see Figure 1). Homomorphisms

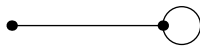


FIGURE 1. H_I

from G to H_I correspond exactly to independent sets of G , as vertices that are mapped to the unlooped vertex of H_I form an independent set, and there is no restriction on the vertices mapped to the looped vertex of H_I . Thus, if we let $\mathcal{I}(G)$ be the set of independent sets of G and $i(G) = |\mathcal{I}(G)|$, then $i(G) = \text{hom}(G, H_I)$. Kahn [8] gave the following upper bound on $i(G)$ when G is an r -regular bipartite graph.

Theorem 1.1. *If G is an r -regular bipartite graph on n vertices, then*

$$i(G) \leq (2^{r+1} - 1)^{n/2r}.$$

Zhao [15] extended the above bound to all r -regular graphs. Also, Galvin and Tetali [7] generalized Theorem 1.1 to general homomorphisms, proving that $\text{hom}(G, H) \leq \text{hom}(K_{r,r}, H)^{n/2r}$ whenever G is a bipartite r -regular graph. For the problem of maximizing the number of independent sets over graphs of a fixed order and size, the solution is a corollary of the Kruskal-Katona theorem [10, 9]. Define the *lex graph* on n vertices and m edges, denoted $L(n, m)$, to be the graph with vertex set $[n]$ and edge set given by the initial m elements in the lexicographic ordering¹ on $\binom{[n]}{2}$. We have the following (see [5] for a proof).

Theorem 1.2. *If G is a graph on n vertices and m edges, then*

$$i(G) \leq i(L(n, m)).$$

The problem of computing the partition function for the Widom-Rowlinson model in statistical physics is another homomorphism enumeration problem. Writing P_3^o for the fully-looped path on three vertices, the number of states of the Widom-Rowlinson model on graph G is exactly $\text{hom}(G, P_3^o)$. In [5], the present authors determined the maximum possible value for $\text{wr}(G) = \text{hom}(G, P_3^o)$. To be precise, for each n and m , a collection of five graphs was exhibited such that at least one of them attains the maximum value of $\text{wr}(G)$ among all graphs with n vertices and m edges.

The image graphs H_I and P_3^o are examples of loop-quasi-threshold graphs. (We define this notion in the following section.) We say that a graph G is $\text{hom}(\cdot, H)$ -*extremal* if $\text{hom}(G, H) \geq \text{hom}(G', H)$ for all graphs G' with $n(G') = n(G)$ and $e(G') = e(G)$. If the graph H is clear from context, we will simply call G extremal. In Section 2, we prove a broad result showing that if H is a loop-quasi-threshold image graph, then there is a quasi-threshold $\text{hom}(\cdot, H)$ -extremal graph G of every order and size. Our approach is based on compressions as in [5]. For most small loop-quasi-threshold graphs, the extremal problem is understood. The analysis of weighted independent sets in [6], together with the results in [5], take care of all connected loop-quasi-threshold image graphs on at most three vertices with one exception. For a comprehensive survey of extremal enumeration problems for small loop-quasi-threshold image graphs, and a result on the one exceptional image graph, see [3].

The latter sections of this paper will investigate homomorphisms into this exceptional graph that we call *the fox*. (It has also been called the wrench by Brightwell and Winkler [1].) The fox, denoted F , is formed from P_3^o by deleting the loop on one of the end-vertices of the path (see Figure 2). Note that elements of $\text{Hom}(G, F)$ have a rather natural graph theoretic interpretation.

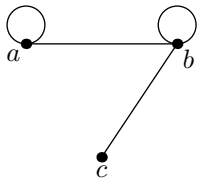


FIGURE 2. The fox, F

Using the labeling of the vertices of F in Figure 2, we note that vertices of G mapped to c form an independent set, and vertices mapped to a and c must be non-adjacent. Thus, in the complement

¹A set A is less than B in the lex order if $\min(A \Delta B) \in A$.

of G , vertices that get mapped to a and c form a split subgraph² of G . So we have that $\text{hom}(G, F)$ counts the number of split subgraphs of \overline{G} . We let $\text{vl}(G) = \text{hom}(G, F)$.

In Sections 3-5, we solve the problem of determining $\max \{\text{vl}(G) : n(G) = n, e(G) = m\}$. It turns out that, in its broad outlines, the behavior is very similar to that of the Widom-Rowlinson problem. In both instances, the extremal graphs transition between “lex-like” and “colex-like” graphs. *A priori*, it was not at all clear that the extremal graphs would exhibit such similar behavior in this case. It would be of interest to determine how widely this phenomenon occurs. Of relevance are the results of Loh, Pikhurko, and Sudakov [12]. They prove that for any fixed image graph H , up to an error factor, the optimizer for $\text{hom}(\cdot, H)$ is a certain $\{0, 1\}$ -valued graphon that is the solution of a quadratically constrained linear program.

The precise form of our result is that we determine, for each n and m , a small set of graphs, one of which must be vl -extremal. In Section 3, we introduce the potentially extremal graphs for $\text{vl}(G)$. In Section 4, we state and prove the main theorem, relying on several technical lemmas, whose proofs are given in Section 5.

2. THRESHOLD AND QUASI-THRESHOLD OPTIMIZERS

In this section we will show that for suitable image graphs H we can guarantee that we can find $\text{hom}(\cdot, H)$ -extremal graphs G that have a structural similarity with H . To be precise we will show that loop-quasi-threshold graphs H have quasi-threshold optimizers G , and loop-threshold H have threshold optimizers G . We define these classes in this section and quote some elementary facts about them. These classes have been extensively studied—see, for instance, [13]. Part of our reason for studying threshold graphs is that the fox is not only loop-quasi-threshold but, in fact, loop-threshold.

As the examples of H_I and F attest, it is natural to consider image graphs with loops. On the other hand, the natural extremal question is which *simple* graph G (with m edges) maximizes $\text{hom}(G, H)$. We therefore need to make a distinction between threshold graphs G , which are simple, and what we call *loop-threshold graphs* H , which may contain loops. We define threshold graphs here and postpone the definition of quasi-threshold graphs.

Definition. A simple graph G is a *threshold graph* if there exists a function $w : V(G) \rightarrow \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that $x \sim_G y$ if and only if $w(x) + w(y) \geq t$. Equivalently (see [2]) a graph is threshold if it can be constructed inductively from a single vertex by successively adding dominating vertices or isolated vertices. The *code* of a threshold graph on vertex set $[n]$ is the sequence $(c_1, c_2, \dots, c_n) \in \{0, 1\}^n$ such that, for $x \neq y$,

$$x \sim_G y \quad \text{if and only if} \quad c_{\min(x,y)} = 1.$$

Similarly, a *loop-threshold graph* is constructed inductively from a single, possibly looped, vertex by successively adding looped dominating vertices or unlooped isolated vertices. Again, we can define the *code* of a loop-threshold graph H on vertex set $[n]$ to be the sequence $(c_1, c_2, \dots, c_n) \in \{0, 1\}^n$ such that

$$x \sim_H y \quad \text{if and only if} \quad c_{\min(x,y)} = 1,$$

where we have allowed for the possibility that $x = y$.

Note that if our code is (c_1, c_2, \dots, c_n) , then whether vertex v_i is adjacent to v_j is determined, if $i < j$, by c_i . In particular, for threshold graphs, the value c_n has no effect on the graph, but for loop-threshold graphs, the value c_n determines whether v_n is looped or not. One can think of

²We define the *join* of graphs G and H , denoted $G \vee H$, to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{x, y\} : x \in V(G) \text{ and } y \in V(H)\}$. A *split graph* is a graph of the form $K_p \vee E_q$ for some integers $p, q \geq 1$.

constructing the graph from right to left: first we put down v_n , then we put down v_{n-1} , joined to v_n or not according to the value of c_{n-1} . We continue in this way, adding vertices and joining them to all or none of the vertices added so far (including the vertex itself in the loop-threshold case).

We will write expressions such as $1^a 0^b 1^c \dots$ for the sequence consisting of a sequence of a 1s, b 0s, c 1s, etc. Zeroes are only adjacent to ones to their left, while ones are adjacent to every vertex on their right and also ones on their left. For instance, the loop-threshold graph with code $1^4 0^6$ is $K_4^o \vee E_6$, where K_4^o is a fully-looped K_4 . Similarly, the fox has code 101. Given sequences σ and τ , we write $\sigma.\tau$ for their concatenation. For a $\{0, 1\}$ -sequence σ , we let $T(\sigma)$ be the threshold graph with code σ . See [5] for a more extensive discussion.

The definition of (loop-)quasi-threshold graphs parallels the inductive definition of threshold graphs.

Definition. A simple graph G is *quasi-threshold* if it is either K_1 , a non-trivial disjoint union of two quasi-threshold graphs, or a join of the form $K_1 \vee G'$ where G' is a quasi-threshold graph. A *loop-quasi-threshold graph* is constructed inductively as either a K_1 (possibly looped), a disjoint union of two loop-quasi-threshold graphs, or the join of a loop-quasi-threshold graph with a looped K_1 .

See Figure 3 for some examples of threshold, loop-threshold, quasi-threshold, and loop-quasi-threshold graphs. Threshold and quasi-threshold graphs have similar characterizations in terms

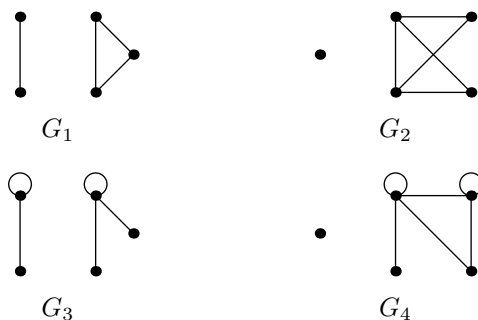


FIGURE 3. Examples: G_1 is quasi-threshold, but not threshold, G_2 is threshold, G_3 is loop-quasi-threshold, but not loop-threshold, and G_4 is loop-threshold.

of neighborhood inclusion. For a vertex v in a graph G , we define the (*open*) *neighborhood* of v , denoted $N(v)$, by $N(v) = \{y \in V(G) : vy \in E(G)\}$.

Lemma 2.1.

(1) A graph G is threshold if for all distinct $x, y \in V(G)$ either

$$N(x) \setminus \{y\} \subseteq N(y) \setminus \{x\} \quad \text{or} \quad N(y) \setminus \{x\} \subseteq N(x) \setminus \{y\}.$$

(2) A graph G is quasi-threshold if for all $x, y \in V(G)$ with $x \sim y$ either

$$N(x) \setminus \{y\} \subseteq N(y) \setminus \{x\} \quad \text{or} \quad N(y) \setminus \{x\} \subseteq N(x) \setminus \{y\}.$$

Proof. See [2] for (1) and [5] for (2). □

Lemma 2.2.

(1) A graph H is loop-threshold if for all $x, y \in V(H)$ either

$$N(x) \subseteq N(y) \quad \text{or} \quad N(y) \subseteq N(x).$$

(2) A graph H is loop-quasi-threshold if for all $x, y \in V(H)$ with $x \sim y$ either

$$N(x) \subseteq N(y) \quad \text{or} \quad N(y) \subseteq N(x).$$

Proof. Straightforward adaptation of the proof of Lemma 2.1. \square

We now introduce a compression which makes a graph “more threshold”. Let G be any non-complete graph, and let x and y be distinct vertices in G . The choice of x and y defines a natural partition of $V(G \setminus \{x, y\})$ into four parts: vertices which are adjacent only to x , vertices adjacent only to y , vertices adjacent to both and vertices adjacent to neither. We write

$$\begin{aligned} A_{x\bar{y}} &= \{v \in V(G \setminus \{x, y\}) : v \sim x, v \not\sim y\}, \\ A_{xy} &= \{v \in V(G \setminus \{x, y\}) : v \sim x, v \sim y\}, \text{ and} \\ A_{\bar{x}y} &= \{v \in V(G \setminus \{x, y\}) : v \not\sim x, v \sim y\}. \end{aligned}$$

Definition. The *compression of G from x to y* , denoted $G_{x \rightarrow y}$, is the graph obtained from G by deleting all edges between x and $A_{x\bar{y}}$ and adding all edges from y to $A_{x\bar{y}}$.

It is relatively straightforward to prove that a graph with the property that no compression changes its isomorphism type is threshold. Rather than prove this explicitly, we find it more convenient in the next lemma to use the degree variance, or equivalently

$$d_2(G) = \sum_{x \in V(G)} d^2(x),$$

as a measure of how close we are to being threshold. The lemma shows that repeatedly applying compression eventually yields a (quasi-)threshold graph.

Lemma 2.3. *Suppose that \mathcal{G} is a family of graphs on a fixed vertex set V .*

(1) *If \mathcal{G} is closed under all compressions, i.e., for any $G' \in \mathcal{G}$ and any $x, y \in V$ we also have $G'_{x \rightarrow y} \in \mathcal{G}$, and moreover G satisfies*

$$d_2(G) = \max \{d_2(G') : G' \in \mathcal{G}\},$$

then G is threshold.

(2) *If \mathcal{G} is closed under adjacent compressions, i.e., for any $G' \in \mathcal{G}$ and any $x, y \in V$ with $x \sim y$ we also have $G'_{x \rightarrow y} \in \mathcal{G}$, and moreover G satisfies*

$$d_2(G) = \max \{d_2(G') : G' \in \mathcal{G}\},$$

then G is quasi-threshold.

Proof. It is easy to see that if $x, y \in G$ have incomparable neighborhoods in the sense of Lemma 2.1, then $d_2(G_{x \rightarrow y}) > d_2(G)$, contradicting our assumption on G . Thus, by Lemma 2.1, G is respectively threshold or quasi-threshold. \square

We are ready to state and prove the main result of this section. This will imply that for every n and m , there is a vl-extremal threshold graph.

Theorem 2.4.

(1) *If H is a loop-threshold graph, $n \geq 0$, and $0 \leq m \leq \binom{n}{2}$, then there is a threshold graph G with n vertices and m edges such that*

$$\text{hom}(G, H) = \max \{ \text{hom}(G', H) : G' \text{ has } n \text{ vertices and } m \text{ edges} \}.$$

- (2) If H is a loop-quasi-threshold graph, $n \geq 0$, and $0 \leq m \leq \binom{n}{2}$, then there is a quasi-threshold graph G with n vertices and m edges such that

$$\text{hom}(G, H) = \max \{ \text{hom}(G', H) : G' \text{ has } n \text{ vertices and } m \text{ edges} \}.$$

Proof. We prove (1), the other case is similar. By Lemma 2.3, it suffices to show that the set of graphs on vertex set $[n]$, having m edges, and maximizing $\text{hom}(\cdot, H)$ is closed under arbitrary compressions. This, in turn, follows if we can show $\text{hom}(G, H) \leq \text{hom}(G_{x \rightarrow y}, H)$ for all G . To see this, we construct an injection

$$\text{Hom}(G, H) \setminus \text{Hom}(G_{x \rightarrow y}, H) \hookrightarrow \text{Hom}(G_{x \rightarrow y}, H) \setminus \text{Hom}(G, H).$$

Given $\phi \in \text{Hom}(G, H) \setminus \text{Hom}(G_{x \rightarrow y}, H)$, we define a new map $\phi' : V(G) \rightarrow V(H)$ that switches the colors of x and y . (Given a homomorphism $\phi \in \text{Hom}(G, H)$ and a vertex v of G , we call $\phi(v)$ the *color* of v , by analogy with the case $H = K_q$.) To be precise,

$$\phi'(w) = \begin{cases} \phi(w) & \text{if } w \neq x, y \\ \phi(x) & \text{if } w = y \\ \phi(y) & \text{if } w = x. \end{cases}$$

It remains to show that $\phi' \in \text{Hom}(G_{x \rightarrow y}, H) \setminus \text{Hom}(G, H)$. Since $\phi \notin \text{Hom}(G_{x \rightarrow y}, H)$, there must exist $z \in A_{x\bar{y}}$ such that $\phi(z) \sim \phi(x)$, but $\phi(z) \not\sim \phi(y)$. This shows that $\phi' \notin \text{Hom}(G, H)$ since $\phi'(z) \not\sim \phi'(x)$. To show $\phi' \in \text{Hom}(G_{x \rightarrow y}, H)$, note that most adjacencies are preserved; we only need to check adjacencies involving x or y . Also, H is loop-threshold and so the existence of z as above implies that $N(\phi(x)) \supsetneq N(\phi(y))$. If $w \in N_{G_{x \rightarrow y}}(y)$, then $w \in N_G(x) \cup N_G(y)$. Hence, $\phi(w)$ is adjacent in H to either $\phi(x)$ or $\phi(y)$, but, by the inclusion above, this implies

$$\phi'(w) = \phi(w) \sim \phi(x) = \phi'(y).$$

On the other hand, if $w \in N_{G_{x \rightarrow y}}(x) \setminus \{y\}$, then $w \in N_G(x) \cap N_G(y)$. Thus, $\phi(w)$ is adjacent in H to both $\phi(x)$ and $\phi(y)$ and in particular $\phi(w) \in N_H(\phi(y))$. So,

$$\phi'(w) = \phi(w) \sim \phi(y) = \phi'(x).$$

Finally, if $x \sim_{G_{x \rightarrow y}} y$, then $x \sim_G y$ so $\phi(x)$ and $\phi(y)$ are adjacent in H , i.e., $\phi'(x) \sim_H \phi'(y)$. \square

3. THE POTENTIALLY EXTREMAL GRAPHS

For the remainder of the paper, we work on the problem of maximizing $\text{vl}(G) = \text{hom}(G, F)$ over graphs on n vertices and m edges. We know, from Theorem 2.4, that we can restrict our attention to threshold graphs. We prove that there are a small number of threshold graphs that are candidates for being the extremal graph. In this section, we introduce these potentially extremal graphs. In the next section, we present a collection of lemmas and deduce our main theorem. In the final section, we prove these lemmas.

All of the potentially extremal graphs look very similar either to the lex graph or to the colex graph. The lex graph was discussed in the introduction, and is very similar to the split graph $\mathcal{S}(n, k) = K_k \vee E_{n-k}$ for some k . The *colex graph* with n vertices and m edges, denoted $\mathcal{C}(n, m)$, has vertex set $[n]$ and its edges are the first m elements of $\binom{[n]}{2}$ in colex order³. All colex graphs are similar to $K_q \cup E_{n-q}$ for some q . In fact, it will be valuable in our analysis to incorporate the appropriate value of k for lex graphs and the appropriate value of q for colex graphs into our notation.

³ A is less than B in the colex order if $\max(A \triangle B) \in B$.

The first three examples are all very close to being split graphs. The first “almost split” graph is the lex graph as defined above. Lex graphs interpolate between $\mathcal{S}(n, k-1)$ and $\mathcal{S}(n, k)$. If

$$\binom{k-1}{2} + (k-1)(n-k+1) \leq \binom{k-1}{2} + (k-1)(n-k+1) + w < \binom{k}{2} + k(n-k),$$

i.e., $0 \leq w \leq n-k-1$, then the lex graph with $e = \binom{k-1}{2} + (k-1)(n-k+1) + w$ edges is

$$\mathcal{L}(n, k, w) := \mathcal{L}(n, e) = \mathcal{S}(n, k) - \{kx : k+w+1 \leq x \leq n\}.$$

Note that $\mathcal{S}(n, k-1) = \mathcal{L}(n, k, 0)$, so that all split graphs are special examples of lex graphs.

Our next potentially extremal graph is the split graph $\mathcal{S}(n-1, k)$ with a vertex adjacent to a portion of the complete subgraph. For $0 \leq w \leq k \leq n$, we define the *deficient split graph* to be

$$\mathcal{S}_0(n, k, w) = \mathcal{S}(n-1, k) + \{nx : 1 \leq x \leq w\}.$$

The *isodeficient split graph* occurs when we add an isolate to a deficient split graph. So, for $0 \leq w \leq k \leq n$, we have

$$\mathcal{S}_1(n, k, w) = \mathcal{S}(n-2, k) + \{(n-1)x : 1 \leq x \leq w\}.$$

The other potentially extremal examples are close to colex graphs. Colex graphs themselves interpolate between graphs

$$\mathcal{U}(n, q) = K_q \cup E_{n-q},$$

with consecutive values of q . To be precise, for $0 \leq w \leq q \leq n-1$,

$$\mathcal{C}(n, q, w) := \mathcal{C}(n, e) = \mathcal{U}(n, q) + \{x(n-q) : n-w+1 \leq x \leq n\},$$

with $e = \binom{q}{2} + w$. Thus, $\mathcal{C}(n, q, w)$ is a complete graph on q vertices, together with one vertex joined to w of these, and $n-q-1$ isolates.

Our final potentially extremal graph is the *antitriangle graph*. This is a $\mathcal{U}(n, q)$ with a triangle deleted. We let

$$\nabla(n, k) = \mathcal{U}(n, q) - \{12, 23, 13\}.$$

Definition. A graph G is said to be *potentially extremal* if

$$G \in \{\mathcal{L}(n, k, w), \mathcal{C}(n, q, w), \nabla(n, q), \mathcal{S}_0(n, k, w), \mathcal{S}_1(n, k, w)\}$$

for some values of n, k, q, w . See Figure 4 for drawings of the potentially extremal graphs.

Lemma 3.1. *The potentially extremal graphs have code and number of edges as given below. The codes specified give graphs isomorphic to those defined above.*

G	Code	Name of code	$e(G)$
$\mathcal{S}(n, k)$	$1^k 0^{n-k}$	$s(n, k)$	$\binom{k}{2} + k(n-k)$
$\mathcal{S}_0(n, k, w)$	$1^w 0 1^{k-w} 0^{n-k-1}$	$s_0(n, k, w)$	$\binom{k}{2} + k(n-k-1) + w$
$\mathcal{S}_1(n, k, w)$	$0 1^w 0 1^{k-w} 0^{n-k-2}$	$s_1(n, k, w)$	$\binom{k}{2} + k(n-k-2) + w$
$\mathcal{L}(n, k, w)$	$1^{k-1} 0^{n-k-w} 1 0^w$	$\ell(n, k, w)$	$\binom{k}{2} + (k-1)(n-k) + w$
$\nabla(n, q)$	$0^{n-q} 1^q 0^3$	$a(n, q)$	$\binom{q}{2} - 3$
$\mathcal{C}(n, q, w)$	$0^{n-q-1} 1^w 0 1^{q-w}$	$c(n, q, w)$	$\binom{q}{2} + w$

Proof. Routine calculation. □

Lemma 3.2.

- (1) If $v \in V(G)$ is an isolated vertex, then $\text{vl}(G) = 3 \text{vl}(G \setminus \{v\})$, and
- (2) If $v \in V(G)$ is a dominating vertex, then $\text{vl}(G) = \text{vl}(G \setminus \{v\}) + 2^{n-1} + 1$.

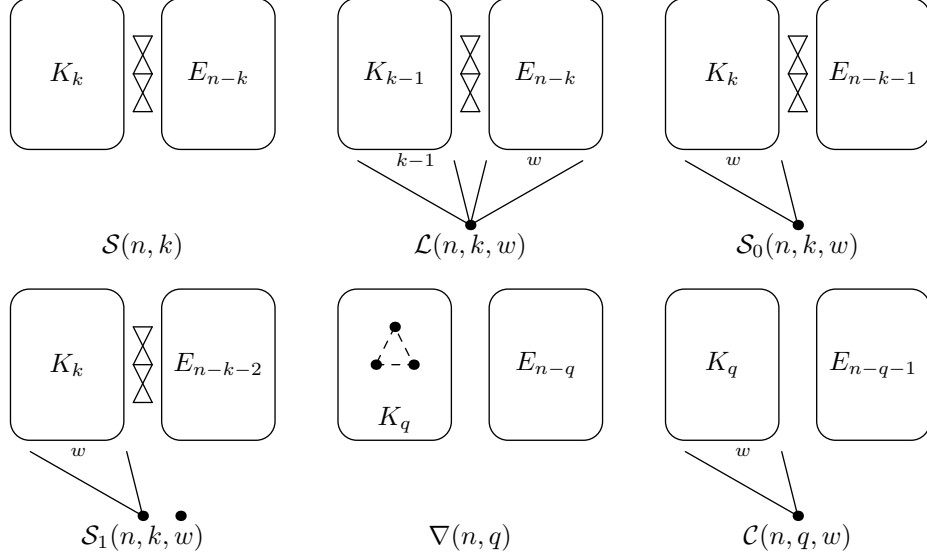


FIGURE 4. The potentially extremal graphs.

Proof. Straightforward. □

Lemma 3.3. *The values of $\text{vl}(G)$ for the potentially extremal graphs G are as follows.*

G	$\text{vl}(G)$
$\mathcal{S}(n, k)$	$3^{n-k} + 2^n - 2^{n-k} + k$
$\mathcal{S}_0(n, k, w)$	$3^{n-k} + 2^n - 2^{n-k} + 2^{n-w-1} - 2^{n-k-1} + k + 2(k-w)$
$\mathcal{S}_1(n, k, w)$	$3(3^{n-1-k} + 2^{n-1} - 2^{n-1-k} + 2^{n-w-2} - 2^{n-k-2} + k + 2(k-w))$
$\mathcal{L}(n, k, w)$	$3^{n-k} + (2^w + 1)3^{n-k-w} + 2^n - 2^{n-k+1} + k - 1$
$\nabla(n, q)$	$3^{n-q} (2^q + q + 16)$
$\mathcal{C}(n, q, w)$	$3^{n-q-1} (2^{q+1} + 2^{q-w} + q + 2(q-w))$

Proof. Routine calculation. □

4. PROOF OF THE MAIN THEOREM

In this section, we will state our main theorem precisely and prove it, relying on a sequence of lemmas whose proofs we delay until the following section.

Theorem 4.1. *Let G be a graph on n vertices and m edges. Then there is some*

$$H \in \{\mathcal{L}(n, k, w), \mathcal{C}(n, q, w), \nabla(n, q), \mathcal{S}_0(n, k, w), \mathcal{S}_1(n, k, w)\}$$

such that $\text{vl}(G) \leq \text{vl}(H)$ and H has m edges (and, of course, n vertices). Moreover, if G is vl-extremal and threshold, then it is isomorphic to one of the graphs above.

Note that if G is vl-extremal, this does not imply that G is potentially extremal. It is possible that there are vl-extremal graphs that are not threshold, although we are not aware of any such examples.

As in the Widom-Rowlinson case, the optimal graph transitions from being “lex-like” when m is small to being “colex-like” when m is large. Using the techniques of Loh, Pikhurko, and Sudakov [12], it is straightforward to determine that this transition occurs when the edge density is

$$\frac{4(\ln^4(3) - 2\ln(2)\ln^3(3) + \ln^2(2)\ln^2(3))}{\ln^4(2) + 4\ln^4(3) - 4\ln^3(2)\ln(3) - 8\ln(2)\ln^3(3) + 8\ln^2(2)\ln^2(3)} \simeq 0.422044.$$

Of course, these techniques are only accurate up to a constant (but extremely large) factor since they rely on the regularity lemma.

The tail of an extremal sequence is extremal, so in proving the theorem our main task is to show that the only possible extensions of smaller extremal graphs are the ones allowed in the statement of the theorem. The following technical lemmas allow us to do this. Recall that for $\{0, 1\}$ -sequences σ and τ , we write $\sigma.\tau$ for their concatenation and $T(\sigma)$ for the threshold graph with code σ .

Lemma 4.2. *If $k \leq (\ln(3/2)/\ln(3))n$, then the threshold graph $G = T(0.\ell(n-1, k, w))$ is not vl-extremal unless one of the following conditions holds:*

- (1) $w = 0$, in which case $G = \mathcal{S}_1(n, k-1, k-1)$ is potentially extremal,
- (2) $k = 1$, in which case $G = \mathcal{L}(n, 1, w)$ is potentially extremal,
- (3) $w = n - k - 1$, in which case $G = \mathcal{S}_1(n, k, k)$ is potentially extremal, or
- (4) $w = n - k - 2$, in which case $G = \mathcal{S}_1(n, k, k-1)$ is potentially extremal.

Lemma 4.3. *If $q \geq \frac{n}{2} - 2$, then the threshold graph $G = T(1.c(n-1, q, w))$ is not vl-extremal unless $q = n - 2$ in which case $G = \mathcal{C}(n, n-1, w+1)$ is potentially extremal.*

Lemma 4.4. *If $2q - \log(q) \geq n$, then $G = T(1.a(n-1, q))$ is not vl-extremal, unless $q = n - 1$ in which case $G = \nabla(n, n)$.*

Lemma 4.5. *If $k \leq (\ln(3/2)/\ln(3))n$, then neither $G_0 = T(0.s_1(n-1, k, w))$ nor $G_1 = T(1.s_1(n-1, k, w))$ is vl-extremal except for the cases $T(0.s_1(n-1, 1, 1)) = \mathcal{L}(n, 1, n-3)$ and $T(1.s_1(n-1, 1, 1)) = \mathcal{S}_0(n, 2, 1)$.*

Lemma 4.6.

- (1) *If $n \geq 200$ and $0.35n < k < n - 4$, then $\mathcal{L}(n, k, w)$ is not vl-extremal.*
- (2) *If $n \geq 200$ and $0.35n < k$, then $\mathcal{S}_1(n, k, w)$ is not vl-extremal.*

Lemma 4.7. *If $k \geq n - 4$, then $G = \mathcal{L}(n, k, w)$ is not vl-extremal unless*

- (1) $e(G) \geq \binom{n}{2} - 2$ in which case G is a colex graph and is vl-extremal, or
- (2) $e(G) = \binom{n}{2} - 3$ in which case G is the extremal graph $\nabla(n, n)$.

Lemma 4.8.

- (1) *If $n \geq 200$ and $0.04n < q < 0.6n$, then neither $\mathcal{C}(n, q, w)$ nor $\nabla(n, q)$ is vl-extremal.*
- (2) *If $n \geq 240$ and $21 \leq q < 0.25n$, then neither $\mathcal{C}(n, q, w)$ nor $\nabla(n, q)$ is vl-extremal.*

Proof of Theorem 4.1. We prove the result by induction on n . All the cases with $n \leq 240$ have been verified by computer search. The search is tremendously simplified by the fact that any suffix of a vl-extremal code must itself be extremal. Our program uses this observation to generate a list of all extremal codes on n vertices from the corresponding list for $n - 1$ vertices (see [4]). Therefore, we may assume $n > 240$.

Without loss of generality, we may assume G is vl-extremal. We are left to show that G is potentially extremal, i.e., it falls into one of the five classes in the statement of the theorem. By Theorem 2.4, we may assume that $G = T(\sigma_1, \sigma_2, \dots, \sigma_n)$ is threshold since the fox, F , is loop-threshold. By Lemma 3.2, $G' = T(\sigma_2, \dots, \sigma_n)$ is also vl-extremal. Thus, we may assume, by induction,

$$G' \in \{\mathcal{L}(n-1, k, w), \mathcal{C}(n-1, q, w), \nabla(n-1, q), \mathcal{S}_0(n-1, k, d), \mathcal{S}_1(n-1, k, d)\}$$

for some suitable values of k, w, q, d . The structure of the proof is outlined in Table 1. We deal with the cases $\sigma_1 = 0, 1$, coupled with the five possibilities for the structure of G' .

To complete the proof, we need only address each ‘‘unless’’ in the Table 1. We deal with each one in turn, going down the table.

		Choices for σ_1	
		0	1
$\mathcal{L}(n-1, k, w)$	Not extremal by Lemma 4.2 unless k is large or $k = 1, n - w - 2$		$G = \mathcal{L}(n, k + 1, w)$
$\mathcal{C}(n-1, q, w)$	$G = \mathcal{C}(n, q, w)$		Not extremal by Lemma 4.3 unless q is small
$\nabla(n-1, q)$	$G = \nabla(n, q)$		Not extremal by Lemma 4.4 unless q is small or $q = n - 1$
$\mathcal{S}_0(n-1, k, w)$	$G = \mathcal{S}_1(n, k, w)$		$G = \mathcal{S}_0(n, k + 1, w + 1)$
$\mathcal{S}_1(n-1, k, w)$	Not extremal by Lemma 4.5 unless $k = w = 1$		Not extremal by Lemma 4.5

TABLE 1. Structure of the proof of Theorem 4.1

$G = T(0.\ell(n-1, k, w))$: Lemma 4.2 does not apply if $k > (\ln(3/2)/\ln(3))n$. If this is the case, then Lemma 4.6 implies that G' was not extremal, unless G' is a colex graph in which case G is also colex and hence potentially extremal. If $k \leq (\ln(3/2)/\ln(3))n$, then Lemma 4.2 tells us that G is potentially extremal.

$G = T(1.c(n-1, q, w))$: Lemma 4.3 does not apply if $q < \frac{n}{2} - 2$. If $21 \leq q < 0.6(n-1)$ then, by Lemma 4.8, G' is not extremal. If $q < 21$, then G consists of some graph on at most $\binom{21}{2} - 1$ vertices, together with a collection of isolated vertices. Therefore, our computer analysis of graphs on at most 240 vertices has already established that no such graph is vl-extremal, except $\mathcal{C}(n-1, 0, w)$ for $w \leq 2$. These two graphs are lex graphs.

$G = T(1.a(n-1, q))$: Lemma 4.4 does not apply if $2q - \log(q) \geq n$ in which case Lemma 4.8 implies that $\nabla(n-1, q)$ is not vl-extremal.

$G = T(0.s_1(n-1, 1, 1))$: As we observe in Lemma 4.5, we have $G = \mathcal{L}(n, 1, n-3)$.

We have established that G is potentially extremal, and so we are done. \square

5. PROOFS OF LEMMATA

Lemma 4.2. *If $k \leq (\ln(3/2)/\ln(3))n$, then the threshold graph $G = T(0.\ell(n-1, k, w))$ is not vl-extremal unless one of the following conditions holds:*

- (1) $w = 0$, in which case $G = \mathcal{S}_1(n, k-1, k-1)$ is potentially extremal,
- (2) $k = 1$, in which case $G = \mathcal{L}(n, 1, w)$ is potentially extremal,
- (3) $w = n - k - 1$, in which case $G = \mathcal{S}_1(n, k, k)$ is potentially extremal, or
- (4) $w = n - k - 2$, in which case $G = \mathcal{S}_1(n, k, k-1)$ is potentially extremal.

Proof. Note that

$$\text{vl}(G) = 3 \left(3^{n-1-k} + (2^w + 1)3^{n-1-k-w} + 2^{n-1} - 2^{n-k} + k - 1 \right).$$

There are three cases in the proof depending on the size of w . In each of the three cases, there is an alternative graph G' with the same number of edges which beats G . We let

$$G' = \begin{cases} \mathcal{L}(n, k-1, w+n-2k+2) & \text{if } 0 \leq w \leq k-2, \\ \mathcal{L}(n, k, w-k+1) & \text{if } k-1 \leq w \leq n-2k-1, \\ \mathcal{S}_1(n, k, w-n+2k+1) & \text{if } n-2k \leq w. \end{cases}$$

In each case, we have $e(G) = e(G')$.

In the first case, when $0 \leq w \leq k - 2$, we have

$$\text{vl}(G') = 3^{n-k+1} + (2^{w+n-2k+2} + 1)3^{k-w-1} + 2^n - 2^{n-k+2} + k - 2.$$

First note that

$$3^{n-k} + (2^w + 1)3^{n-k-w} + 3k - 3 \leq 2 \cdot 3^{n-k} + k - 2,$$

since $w \geq 1$ and the linear terms are dominated by the exponential ones. We now need to show

$$3(2^{n-1} - 2^{n-k}) \leq 3^{n-k} + 2^n - 2^{n-k+2},$$

which is equivalent to

$$2^{n-1} + 2^{n-k} \leq 3^{n-k}.$$

This holds provided

$$k \leq \frac{\ln(3/2)}{\ln(3)}n.$$

In the second case, when $k - 1 \leq w \leq n - 2k - 1$, we have

$$\text{vl}(G') = 3^{n-k} + (2^{w-k+1} + 1)3^{n-w-1} + 2^n - 2^{n-k+1} + k - 1.$$

We first deal with the case when $k = 2$, where

$$\begin{aligned} \text{vl}(G') - \text{vl}(G) &= 2^{w-1}3^{n-w-1} + 3^{n-w-1} + 2^{n-1} + 1 - 2^w3^{n-w-2} - 3^{n-w-2} - 3 \cdot 2^{n-2} - 3 \\ &= 2^{w-1}3^{n-w-2} + 2 \cdot 3^{n-w-2} - 2^{n-2} - 2 \\ &> 0. \end{aligned}$$

If $k \geq 3$, then we observe that the dominant (non-canceling) term in $\text{vl}(G')$ is $2^{w-k+1}3^{n-w-1}$. We will use half of this to beat the term $(2^w + 1)3^{n-k-w}$ in $\text{vl}(G)$, and the other half to beat the remaining terms. Clearly, $2^{w-k}3^{n-w-1} \geq (2^w + 1)3^{n-k-w}$. On the other hand,

$$\begin{aligned} 2^{w-k}3^{n-w-1} &= 3^n \left(\frac{2}{3}\right)^{w+1} \left(\frac{1}{2}\right)^{k+1} \\ &\geq 3^n \left(\frac{2}{3}\right)^{n-2k} \left(\frac{1}{2}\right)^{k+1} \\ &= 2^{n-1} \left(\frac{9}{8}\right)^k. \end{aligned}$$

Thus,

$$2^{w-k}3^{n-w-1} + 2^n - 2^{n-k+1} + k - 1 \geq 3 \cdot 2^{n-1} + 3k - 3.$$

In the final case, when $n - 2k \leq w$, we have

$$\text{vl}(G') = 3 \left(3^{n-k-1} + 2^{n-1} - 2^{n-1-k} + 2^{2n-w-2k-3} - 2^{n-k-2} + 3k - 2w \right).$$

We will first do the case $w = n - k - 3$, noting that the case $w = n - k - 2$ is one of the exceptional cases in the lemma. We have

$$\frac{1}{3}(\text{vl}(G') - \text{vl}(G)) = 2^{n-k-3}(8 - 4 - 2) + 4k - 2n + 6 - 2^{n-k-3}(9 - 8) - k - 8 > 0.$$

If $w \leq n - k - 4$, we have $2^{2n-w-2k-3} \geq 2^w3^{n-k-1-w}$ since

$$\frac{2^{2n-w-2k-3}}{2^w3^{n-k-1-w}} = \frac{3}{8} \left(\frac{4}{3}\right)^{n-k-w} \geq \frac{3}{8} \left(\frac{4}{3}\right)^4 = \frac{32}{27}.$$

For the remaining terms, we see

$$2^{n-k} - k + 1 \geq 3 \cdot 2^{n-k-2} + 3k - 2w.$$

□

Lemma 4.3. *If $q \geq \frac{n}{2} - 2$, then the threshold graph $G = T(1.c(n-1, q, w))$ is not vl-extremal unless $q = n - 2$ in which case $G = \mathcal{C}(n, n-1, w+1)$ is potentially extremal.*

Proof. There are two cases in the proof. In each, we compare G to a colex graph $G' = \mathcal{C}(n, q', w')$. In the first case $q' = q + 1$, and in the second $q' = q + 2$.

The first case applies when $w \leq 2q - n + 2$. We set $q' = q + 1$ and $w' = w + n - q - 1$. Note that $w' \leq q + 1$ by our assumption on w . We have

$$\begin{aligned} \text{vl}(G) &= 3^{n-q-2} (2^{q+1} + 2^{q-w} + 3q - 2w) + 2^{n-1} + 1, \text{ and} \\ \text{vl}(G') &= 3^{n-q-2} (2^{q+2} + 2^{2q+2-w-n} + 5q + 5 - 2w - 2n). \end{aligned}$$

So,

$$\begin{aligned} \text{vl}(G') - \text{vl}(G) &\geq [3^{n-q-2} 2^{q+2} - 3^{n-q-2} 2^{q+1} - 3^{n-q-2} 2^{q-w} - 2^{n-1}] - 3^{n-q-2} (3q - 2w) - 1 \\ &= 3^{n-q-2} 2^{q+1} \left[1 - \left(\frac{1}{2}\right)^{w+1} - \left(\frac{2}{3}\right)^{n-q-2} \right] - 3^{n-q-2} (3q - 2w) - 1 \\ &> 0, \end{aligned}$$

provided $q \leq n - 4$. The remaining cases are when $q = n - 2$, which is one of the exceptional cases in the lemma, and $q = n - 3$, which we do separately. We have

$$\begin{aligned} \text{vl}(G') - \text{vl}(G) &= 3 (2^{n-1} + 2^{n-w-4} + 3n - 2w - 15) - 3 (2^{n-2} + 2^{n-w-3} + 3n - 2w - 9) - 2^{n-1} - 1 \\ &= 2^{n-2} - 3 \cdot 2^{n-w-4} - 19 \\ &> 0. \end{aligned}$$

The second case applies when $w > 2q - n + 2$. We set $q' = q + 2$ and $w' = w + n - 2q - 2$. Note that $0 \leq w' \leq q + 2$. Now we have

$$\text{vl}(G') = 3^{n-q-3} (2^{q+3} + 2^{3q+4-w-n} + 7q + 10 - 2w - 2n).$$

When $q \leq n - 5$, we see $3^{n-q-2} 2^{q-w} \leq 3^{n-q-3} 2^{q'-w'}$ since $q' - w' \geq (q - w) + 2$. For the remaining exponential terms, we have

$$3^{n-q-3} 2^{q+3} - 3^{n-q-2} 2^{q+1} - 2^{n-1} = 3^{n-q-3} 2^{q+1} \left[4 - 3 - 3 \left(\frac{2}{3}\right)^{n-q-2} \right] \geq 3^{n-q-3} 2^{q+1} \left(\frac{1}{9}\right),$$

since $n - q - 2 \geq 3$. The remaining terms in $\text{vl}(G') - \text{vl}(G)$ are dominated by this one. The cases $q = n - 4$ and $q = n - 3$ proceed similarly to the above. □

Lemma 4.4. *If $2q - \log(q) \geq n$, then $G = T(1.a(n-1, q))$ is not vl-extremal, unless $q = n - 1$ in which case $G = \nabla(n, n)$.*

Proof. We compare G to $C := \mathcal{C}(n, q + 1, n - q - 4)$, since both graphs have $\binom{q}{2} - 3$ edges. Since $2q - \log(q) \geq n$, it is the case that

$$2^{2q-n+5} + q + 1 + 2(2q - n + 5) \geq 32q + q + 1 + 2(2q - n + 5) \geq 3q + 48,$$

provided $q \geq 2$. Thus, we have

$$\begin{aligned} \text{vl}(C) - \text{vl}(G) &= 3^{n-q-2} (2^{q+2} + 2^{2q-n+5} + q + 1 + 2(2q - n + 5)) \\ &\quad - 3^{n-q-1} (2^q + q + 16) - 2^{n-1} - 1 \\ &\geq 3^{n-q-2} 2^q - 2^{n-1} - 1 \\ &> 0, \end{aligned}$$

as long as $q \leq n - 4$. The cases $q = n - 2$ and $q = n - 3$ are routine. We compare G to the corresponding colex graph. \square

Lemma 4.5. *If $k \leq (\ln(3/2)/\ln(3))n$, then neither $G_0 = T(0.s_1(n-1, k, w))$ nor $G_1 = T(1.s_1(n-1, k, w))$ is vl-extremal except for the cases $T(0.s_1(n-1, 1, 1)) = \mathcal{L}(n, 1, n-3)$ and $T(1.s_1(n-1, 1, 1)) = \mathcal{S}_0(n, 2, 1)$.*

Proof. We may assume that $k \geq 2$. We first deal with the G_0 case. Depending on the value of w , we will compare G_0 to either $G' = \mathcal{L}(n, k, w')$ or $G'' = \mathcal{L}(n, k-1, w'')$. If $w \geq 4k - n$, we use the former; otherwise we use the latter. We start with the case $w \geq 4k - n$, and set $w' = w + n - 4k$. Note that $0 \leq w' \leq n - k$. We have

$$\begin{aligned} \text{vl}(G_0) &= 9 \left(3^{n-2-k} + 2^{n-2} - 2^{n-2-k} + 2^{n-w-3} - 2^{n-k-3} + 3k - 2w \right), \text{ and} \\ \text{vl}(G') &= 3^{n-k} + \left(2^{w+n-4k} + 1 \right) 3^{3k-w} + 2^n - 2^{n-k+1} + k - 1. \end{aligned}$$

Under the conditions of the lemma,

$$\begin{aligned} 2^{n+w-4k} 3^{3k-w} &= 16 \cdot 2^{n-4} \left(\frac{27}{16} \right)^k \left(\frac{2}{3} \right)^w \\ &\geq 16 \cdot 2^{n-4} \left(\frac{27}{16} \right)^{k-w} \left(\frac{9}{8} \right)^w \\ &\geq 34 \cdot 2^{n-4}, \end{aligned}$$

provided $k \geq w + 2$ or $w \geq 2$. Assuming $k \geq 3$, we have

$$\begin{aligned} \text{vl}(G') - \text{vl}(G_0) &\geq 34 \cdot 2^{n-4} + 2^n - 2^{n-k+1} - 9 \cdot 2^{n-2} - 9 \cdot 2^{n-w-3} - 26k \\ &\geq 2^{n-4} (34 + 16 - 4 - 36 - 9) - 26k \\ &> 0. \end{aligned}$$

The remaining case (when $k = 2$) is an easy calculation.

On the other hand, when $1 \leq w \leq 4k - n - 1$, we let $G'' = \mathcal{L}(n, k-1, w'')$ where $w'' = w + 2n - 5k + 1$. Note that $0 \leq w'' \leq n - k + 1$. We have

$$\begin{aligned} \text{vl}(G'') &= 3^{n-k+1} + (2^{w+2n-5k+1} + 1) 3^{4k-n-w-1} + 2^n - 2^{n-k+2} \\ &\geq 3^{n-k+1} + 2^{n-k} + 2^n - 2^{n-k+2}, \end{aligned}$$

since $w \leq 4k - n - 1$. On the other hand, noting that $w \geq 1$,

$$\text{vl}(G_0) \geq 3^{n-k} + 9 \cdot 2^{n-2} - 27 \cdot 2^{n-k-3} + 9 \cdot 2^{n-4}.$$

Thus,

$$\begin{aligned} \text{vl}(G'') - \text{vl}(G_0) &\geq 2 \cdot 3^{n-k} - 29 \cdot 2^{n-4} \\ &> 0, \end{aligned}$$

since by hypothesis $k \leq (\ln(3/2)/\ln(3))n$.

For the G_1 case, we let $S = \mathcal{S}_1(n, k+1, w+1)$. Then,

$$\text{vl}(S) - \text{vl}(G_1) = 2^{n-2} + 2 > 0.$$

□

Lemma 4.6.

- (1) If $n \geq 200$ and $0.35n < k < n - 4$, then $\mathcal{L}(n, k, w)$ is not vl-extremal.
- (2) If $n \geq 200$ and $0.35n < k$, then $\mathcal{S}_1(n, k, w)$ is not vl-extremal.

Proof. Let $L = \mathcal{L}(n, k, w)$ and let C be the colex graph with the same number of vertices and edges as L . We have

$$C \subseteq K_m \cup E_{n-m},$$

where $m = \left\lceil \sqrt{k(n-k)} + \frac{1}{2} \right\rceil$. Also, $\mathcal{S}(n, k-1) \subseteq L$. Hence we have

$$\text{vl}(L) \leq \text{vl}(\mathcal{S}(n, k-1)) \quad \text{and} \quad \text{vl}(C) \geq \text{vl}(K_m \cup E_{n-m}),$$

and it suffices to show $\text{vl}(K_m \cup E_{n-m}) \geq \text{vl}(\mathcal{S}(n, k-1))$. We have

$$\text{vl}(\mathcal{S}(n, k-1)) \leq 3^{n-k+1} + 2^n + k - 1 \leq 3^{n-k+1} + 2^{n+1},$$

whereas

$$\text{vl}(K_m \cup E_{n-m}) \geq 2^m 3^{n-m}.$$

In [5], Lemma 6.6, it is proved that $2^m 3^{n-m} > 3^{n-k+1} + 2^{n+1}$ under the conditions in the lemma. The other case is proved similarly, noting that $\text{vl}(\mathcal{S}_1(n, k, w))$ is at most $9 \text{vl}(\mathcal{S}(n-2, k))$. □

Lemma 4.7. *If $k \geq n - 4$, then $G = \mathcal{L}(n, k, w)$ is not vl-extremal unless*

- (1) $e(G) \geq \binom{n}{2} - 2$ in which case G is a colex graph and is vl-extremal, or
- (2) $e(G) = \binom{n}{2} - 3$ in which case G is the extremal graph $\nabla(n, n)$.

Proof. Since G has so many edges, it will be simpler to analyze $\text{vl}(G)$ in terms of its complementary graph \overline{G} . We split $\text{vl}(G)$ into three terms: one counting homomorphisms which only use labels b and c , and have at least two c 's (see Figure 2 for our labeling of the vertices of F); one counting those in which both a 's and c 's appear; and finally the remainder, those using entirely a 's and b 's, together with those using only b 's and at most one c . The importance of this split is that it is only the final term that depends on n . We have

$$\text{vl}(G) = k(\overline{G}) + \text{spl}(\overline{G}) + (2^n + n),$$

where $k(\overline{G})$ is the number of complete subgraphs of \overline{G} with at least two vertices; $\text{spl}(\overline{G})$ is the number of colorings of $V(\overline{G})$ with colors a , b , and c such that the graph induced by the vertices of colors a and c forms a split graph with the complete side colored c ; and the term $2^n + n$ counts the final class of homomorphisms. Note that $e(\overline{G}) \leq 10$ and adding isolated vertices changes neither $k(\overline{G})$ nor $\text{spl}(\overline{G})$. Thus, it is a simple case analysis to verify all the claims of lemma. □

Lemma 4.8.

- (1) If $n \geq 200$ and $0.04n < q < 0.6n$, then neither $\mathcal{C}(n, q, w)$ nor $\nabla(n, q)$ is vl-extremal.
- (2) If $n \geq 240$ and $21 \leq q < 0.25n$, then neither $\mathcal{C}(n, q, w)$ nor $\nabla(n, q)$ is vl-extremal.

Proof. In both cases, we prove that $G = \mathcal{C}(n, q, w)$ is beaten by a lex graph. Since $K_q \cup E_{n-q}$ is a subgraph of G , we know $\text{vl}(G) \leq \text{vl}(K_q \cup E_{n-q})$. We will choose k in such a way that $S = \mathcal{S}(n, k)$ has $e(S) \geq e(K_q \cup E_{n-q})$ and $\text{vl}(S) > \text{vl}(K_q \cup E_{n-q})$. Indeed, all we will use about $\text{vl}(S)$ is that it is at least 3^{n-k} , and the only bound on $\text{vl}(K_q \cup E_{n-q})$ we use is $\text{vl}(K_q \cup E_{n-q}) \leq 3^{n-q} 2^{q+1}$.

For (1) we define

$$k = \left\lceil n - \sqrt{n^2 - q^2} + \frac{n}{2\sqrt{n^2 - q^2}} \right\rceil.$$

Exactly as in the proof of Lemma 6.8 of [5], this k satisfies our conditions⁴.

For (2) we let $\varepsilon = e(G)/n^2$ and

$$k = \left\lceil \frac{\varepsilon n}{0.85} \right\rceil.$$

The proof now is precisely the same as that of Lemma 6.9 of [5]. The case where $G = \nabla(n, q)$ is essentially identical. \square

6. FURTHER DIRECTIONS

While this paper answers the last remaining extremal question for quasi-loop-threshold image graphs on three or fewer vertices, many interesting questions remain. One, as mentioned above, is to determine whether there are non-threshold extremal graphs. Another would be to solve the extremal homomorphism enumeration question for connected non-loop-quasi-threshold image graphs on three or fewer vertices. (See Figure 5 for a list of such image graphs.) It would seem that

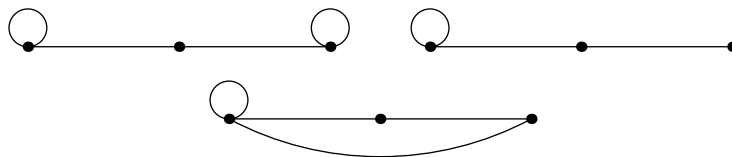


FIGURE 5. The connected image graphs on three or less vertices for which the extremal homomorphism enumeration question remains open.

entirely different techniques need to be developed in order to solve the question for these image graphs.

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⁴That proof referred to the graph parameter $wr(G)$, which satisfies the same inequalities that we are using about $vl(G)$.

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