TREES THROUGH SPECIFIED VERTICES

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Abstract. We prove a conjecture of Horak that can be thought of as an extension of classical results including Dirac's theorem on the existence of Hamiltonian cycles. Namely, we prove for $1 \leq k \leq n-2$ if G is a connected graph with $A \subset V(G)$ such that $d_G(v) \geq k$ for all $v \in A$, then there exists a subtree T of G such that $V(T) \supset A$ and $d_T(v) \leq \lceil \frac{n-1}{k} \rceil$ for all $v \in A$.

1. Introduction

Classical results in Hamiltonian graph theory give sufficient conditions for the existence of a Hamiltonian cycle involving the minimum degree. The starting point for these results is the theorem of Dirac [4].

Theorem 1. *If G is a graph on n vertices such that $n \geq 3$ and for all $v \in V(G)$, $d_G(v) \geq n/2$, then G contains a Hamiltonian cycle.*

Ore [7] showed that the uniform bound can be relaxed and used a condition on the sum of degrees over pairs of nonadjacent vertices.

Theorem 2. *If G is a graph of order $n \geq 3$ such that for any nonadjacent vertices x and y, $d_G(x) + d_G(y) \geq n$, then G is Hamiltonian.*

An extension of this theorem was obtained by Pósa [8].

Theorem 3. *Let G be a connected graph of order $n \geq 3$ such that for any two non-adjacent vertices x and y we have*

$$d_G(x) + d_G(y) \geq k.$$

If $k = n$ then G is Hamiltonian, and if $k < n$ then G contains a path of length k and a cycle of length at least $(k+2)/2$.

The subject of this paper is closely related to a result of Bollobás and Brightwell [2] concerning cycles through vertices with specified degrees. This generalized a question of Katchalski, who asked if we are given a set of vertices with a certain minimum degree condition, when can we find a cycle through a specified number of them? This problem was investigated in the case when the set of vertices was all of $V(G)$ by Alon [1], Egawa and Miyamoto [5] and Bollobás and Häggkvist [3].

Theorem 4. *Let G be a graph with n vertices and let W be a set of $w \geq 3$ vertices of degree at least $d \geq 1$. Moreover, assume that $t := \lceil w/(\lceil n/d \rceil - 1) \rceil \geq 3$. Then there is a cycle in G passing through at least t vertices of W.*

In a very general sense, these classical results in graph theory can be thought of as finding structures with certain maximum degree (in these cases, two) through specified vertices with minimum degree in a graph. The main result of this paper involves finding a subtree through a specified set of vertices in a graph. Further, this subtree has a maximum degree which depends on the minimum degree of the specified vertices.

Throughout we assume that G is a simple graph, and for $v \in V(G)$, we write $d_G(v)$ for the degree of v in G. Also, let the neighborhood of $v \in V(G)$, denoted $N_G(v)$, be defined as $\{u \in V(G) : uv \in$

$E(G)\}$. Lastly, for a set $A \subset V(G)$, let $\Delta_G(A) = \max_{v \in A} d_G(v)$ and likewise $\delta_G(A) = \min_{v \in A} d_G(v)$. The following conjecture was given in a presentation by Horak [6], although it may be due to Stacho.

Conjecture 5. *Let k be such that $1 \leq k \leq n-2$. Given a connected graph G of order n and a set $A \subseteq V(G)$ with $\delta_G(A) \geq k$, there exists a subtree T of G such that $A \subseteq V(T)$ and $\Delta_T(A) \leq \lceil \frac{n-1}{k} \rceil$.*

Thus, Conjecture 5 can be thought of as an extension of several classical theorems of graph theory. It is related to classical questions involving finding structures of bounded degree in certain graphs.

Note that, if true, the conjecture is sharp. To see this, let $k \leq n/2$ and consider the complete bipartite graph $K_{k,n-k}$ as our graph G in the theorem and let $A = V(K_{k,n-k})$. Let M and N be the parts of the $K_{k,n-k}$, with $|M| = k$ and $|N| = n-k$. Then $d_G(v) \geq k$ for all vertices in G. As there are $n-1$ edges in a tree on n vertices and, since all of them must be incident to a vertex of M, $d_T(v) \geq (n-1)/k$ for all $v \in M$. Also, since $d_T(v)$ is integral, we see that $d_T(v) \geq \lceil \frac{n-1}{k} \rceil$ for all $v \in M$.

Also, note that Theorem 4 implies the conjecture is true for $\frac{n}{2} \leq k \leq n-2$, since the maximum degree in the tree must be two, i.e., the tree is a path and we can delete any edge of the cycle to get this path. We further note that the conjecture is nearly true for $k = n-1$. In fact, if $A \neq V(G)$, then the conjecture is true. For if $x \in V(G) \setminus A$, then every vertex in A is adjacent to x, as every vertex in A is adjacent to every vertex in G. Then a star with center x and leaves all vertices in A satisfies the conditions of the conjecture. However, if $V(G) = A$, then $G = K_n$ and we must settle for a Hamiltonian path, meaning the maximum tree degree is 2.

The aim of this paper is to prove Conjecture 5, i.e., the following theorem.

Theorem 6. *Let k be such that $1 \leq k \leq n-2$. Given a connected graph G of order n and a set $A \subseteq V(G)$ with $\delta_G(A) \geq k$, there exists a subtree T of G such that $A \subseteq V(T)$ and $\Delta_T(A) \leq \lceil \frac{n-1}{k} \rceil$.*

It should be noted that the proof is quite a bit more straightforward in the case when $k \leq \sqrt{n-1}$. In fact, the author originally thought that the proof split into two cases depending on whether $k \leq \sqrt{n-1}$, but the proof below, while more complicated, covers both cases.

2. The proof

The aim of this section is to prove Theorem 6. Note first that the result is trivial for $k = 1$, as G is connected and thus contains a spanning tree, and any tree on n vertices has maximum degree at most $n-1$. We may also assume $n \geq 2$. Lastly, since, as noted above, Theorem 4 implies Theorem 6 when $\lceil \frac{n-1}{k} \rceil = 2$, we may assume that $\lceil \frac{n-1}{k} \rceil \geq 3$, or $k < (n-1)/2$.

For $2 \leq k < (n-1)/2$, we use proof by contradiction to get the desired result. Thus we assume that G is a counterexample to the theorem, i.e., G contains a subset of vertices A_0 with $d_G(v) \geq k$ for all vertices $v \in A_0$ and every tree T that contains A_0 in G has a vertex in A_0 of degree greater than $\lceil (n-1)/k \rceil$. In fact, amongst all counterexamples, we shall consider one with the minimal number of edges. We consider trees in G containing all vertices of A_0 that have the smallest maximum degree of vertices in A_0, say Δ. Let $\mathcal{T}$ be the set of such trees with the minimal number of vertices of A_0 of degree Δ and let $T \in \mathcal{T}$.

Let $x_0 \in A_0$ be a vertex of maximum degree in T, i.e., $d_T(x_0) = \Delta = \Delta_T(A_0) > \lceil \frac{n-1}{k} \rceil$. Throughout the proof, we shall use $N_G(x)$ to denote the neighborhood of x in G; similarly, $N_T(x)$ denotes the T-neighborhood of x. Further, if $S \subset V(G)$, $N_G(S) = \cup_{s \in S} N_G(s)$. So, we know that $|N_T(x_0)| = \Delta > \lceil \frac{n-1}{k} \rceil$ and, by our note above, we may assume $\Delta \geq 4$.

The basic idea of the proof is to first construct a modified neighborhood of x_0 with some helpful properties, including being contained in A_0. This allows us to get some bound on the number of edges between this modified neighborhood and the vertices not lying on paths in T back to x_0 from the neighborhood. This bound, in turn, allows us to count vertices outside of the neighborhood which leads to the desired contradiction.

Now, for some $T_0 \in \mathcal{T}$, we would like to find a particular subtree of T_0. The key property of the subtree is that the leaves have no edges between them. We shall consider the subset of $\mathcal{T}$, say $\mathcal{T}_{x_0}$, of all trees $T' \in \mathcal{T}$ such that $d_{T'}(x_0) = \Delta$. We will think of trees in $\mathcal{T}_{x_0}$ as rooted at x_0 and will refer to predecessors of a vertex v, meaning the vertices lying on the path in T from v to x_0. We then build the subtree in stages, depending on the distance from x_0. For a tree T, $v \in V(T)$ and $X \subset V(T)$, let $T\langle v, X\rangle$ subtree of T formed by taking the union of all paths from v to any vertex in X. In fact, for the lemma to go through, we need to consider the subset of $\mathcal{T}_{x_0}$ consisting of edge-minimal trees, say $\mathcal{T}_{x_0}^{\min}$.

Lemma 1. *There is a subtree $\mathbb{T}$ of some $T_0 \in \mathcal{T}_{x_0}^{\min}$ such that*

(1) *$\mathbb{T}$ has at least $\Delta \geq \left\lceil \frac{n-1}{k} \right\rceil + 1$ leaves, the set of which will be denoted $\mathcal{L} = \mathcal{L}(\mathbb{T})$,*
(2) *$\mathcal{L} \subset A_0$,*
(3) *for all $\ell \in \mathcal{L}$, $|N_G(\ell) \cap \mathbb{T}| = 1$, and*
(4) *for all $\ell \in \mathcal{L}$, $d_{T_0}(\ell) < \Delta - 1$.*

Proof. We begin by considering an arbitrary $T \in \mathcal{T}_{x_0}^{\min}$. Let $N^1(T)$ be the set of vertices of A_0 encountered first on any path any from x_0, i.e., the members of A_0 which have no predecessors in A_0 other than x_0. Note that since T is edge-minimal, every path from x_0 reaches a member of A_0 eventually. We then let $\mathcal{T}_{x_0}^1$ be the set of trees T in $\mathcal{T}_{x_0}^{\min}$ with the number of edges in $T\langle x_0, N^1(T)\rangle$ minimized (note that in this process, $N^1(T)$ may change from tree to tree). Let $T^1 \in \mathcal{T}_{x_0}^1$. We begin by showing that if there is an edge of $G - T^1$ between a vertex v of $N^1(T^1)$ and any other vertex of $T^1\langle x_0, N^1(T^1)\rangle$, then $d_{T^1}(v) \geq \Delta - 1$.

Let C^1 be the subset of N^1 which consists of vertices which are adjacent in G to a vertex in $T^1\langle x_0, N^1(T^1)\rangle$ other than its T^1-parent. We will show that for any $v \in C^1$, $d_T(v) \geq \Delta - 1$. Let $v \in C^1$ and suppose that v is adjacent in G to $w \in T^1\langle x_0, N^1(T^1)\rangle$, where w is not the parent of v in T^1. We split the argument into two cases, depending on whether w is a predecessor of v or not. Firstly, suppose w is a predecessor of v. In this case, we know that $\mathrm{dist}_{T^1}(v, w) \geq 2$ since v and w are adjacent in $G - T^1$. Then if we add vw to T^1, and remove the first edge on a path in T^1 from w to v, we have decreased the number of edges in $T^1\langle x_0, N^1(T^1)\rangle$, a contradiction. We can do this without creating a new vertex of T^1-degree Δ (or even $\Delta + 1$) unless $d_{T^1}(v) \geq \Delta - 1$. Secondly, if w is not a predecessor of v, then we add vw to T^1 and remove the first edge on a path in T^1 from w to x_0. Then we have either reduced the degree of x_0 (if $wx_0 \in T^1$) or this gives a tree $\bar{T}^1$ with fewer edges in $\bar{T}^1\langle x_0, N^1(\bar{T}^1)\rangle$ than in $T^1\langle x_0, N^1(T^1)\rangle$, contradicting the assumption that $T^1 \in \mathcal{T}_{x_0}^1$. Once again, we can do this without creating a new vertex of degree Δ or $\Delta + 1$ unless $d_{T^1}(v) \geq \Delta - 1$.

So, we have shown now that if $v \in C^1$, then $d_{T^1}(v) \geq \Delta - 1$. Note that there could be other vertices in N^1 which have T^1-degree at least $\Delta - 1$. Let the set of vertices $v \in N^1 \setminus C^1$ such that $d_{T^1}(v) \geq \Delta - 1$ be denoted D^1 and $B^1 = C^1 \uplus D^1$ (See Figure 1). Thus, B^1 is the set of vertices $v \in N^1$ with T^1-degree at least $\Delta - 1$. At this point, for reasons that will become apparent in a bit, we let $\mathcal{B}_{x_0}^1$ be the subset of $\mathcal{T}_{x_0}^1$ such that $|B^1|$ is minimized. If there exists a $T \in \mathcal{B}_{x_0}^1$ such that $B^1 = \emptyset$, we are done as the conditions of the lemma are satisfied by $\mathbb{T} = T^1\langle x_0, N^1\rangle$. If not, then for each $v \in B^1$, we consider the set of members of A_0 which are first encountered on a path from v in T^1 away from x_0. Let the set of elements of A_0 reached in this way be N_v^2. We then let $N^2(T^1) = (N^1 \setminus B^1) \cup (\cup_{v \in B^1} N_v^2)$. Also, let $\mathcal{T}_{x_0}^2$ be the set of trees $T \in \mathcal{B}_{x_0}^1$ with the number of edges in $T\langle x_0, N^2(T)\rangle$ minimized.

Working in the general case, let $\mathcal{T}_{x_0}^i$ be the set of trees $T^i \in \mathcal{B}_{x_0}^{i-1}$ so that the number of edges in $T^i\langle x_0, N^i(T^i)\rangle$ is minimized. Note that when $T^i \in \mathcal{T}_{x_0}^i$, then $|E(T^i\langle x_0, N^i(T^i)\rangle)| >$

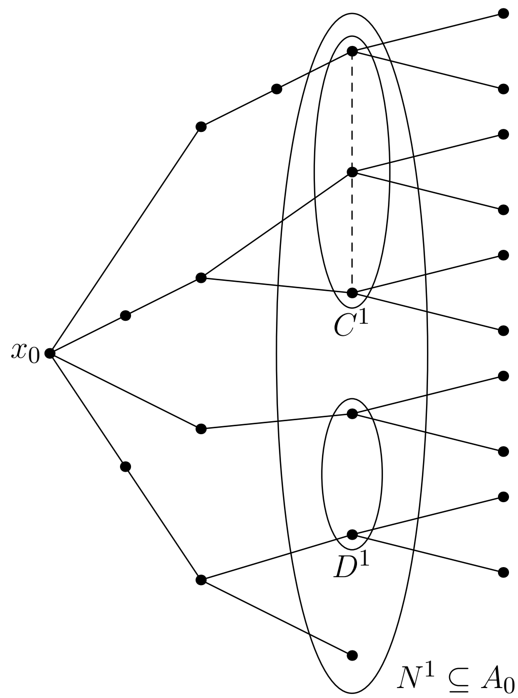

FIGURE 1. An example of N^1, C^1 and D^1 with $\Delta = 4$. Solid lines represent edges in the tree, while dotted edges are in G but not the tree.

$|E(T^i\langle x_0, N^{i-1}(T^i)\rangle)|$ since otherwise we could find a subtree $\bar{T}^i$ of T^i with fewer edges in $\bar{T}^i\langle x_0, N^{i-1}(\bar{T}^i)\rangle$ than in $T^i\langle x_0, N^{i-1}(T^i)\rangle$, a contradiction. Thus,

$$\begin{aligned} |V(T^i\langle x_0, N^i(T^i)\rangle)| &= |E(T^i\langle x_0, N^i(T^i)\rangle)| + 1 \\ &> |E(T^i\langle x_0, N^{i-1}(T^i)\rangle)| + 1 \\ &= |V(T^i\langle x_0, N^{i-1}(T^i)\rangle)|. \end{aligned} \tag{1}$$

Now, let C^i be the subset of N^i consisting of vertices that are adjacent in G to a vertex in $T^i\langle x_0, N^i(T^i)\rangle$ other than its parent. We must now use a bit more care when showing that if $v \in C^i$, then $d_{T^i}(v)$ is large. Whereas in the initial case, we could show that if $y \in C^1$, then $d_{T^1}(y) \geq \Delta - 1$, we can now only show that if $v \in C^i$, then $d_{T^i}(v) \geq \Delta - 2$. This will, however, suffice to give us our desired tree. To this end, let $v \in C^i$ and let w be the vertex in $T^i\langle x_0, N^i(T^i)\rangle$ to which v is adjacent in G, other than its parent.

If w is not a predecessor of v, then we can add vw to T^i and delete the first edge on the path in T^i from w to x_0. If $d_{T^i}(v) < \Delta - 1$ in this case, this operation reduces the number of edges in $T^i\langle x_0, N^j(T^i)\rangle$ for some $j \leq i$, a contradiction to the assumption that $T^i \in \mathcal{T}^i_{x_0}$. On the other hand, if w is a predecessor of v, then the proof splits into two cases. Firstly, if w is not a parent of a vertex of $N^j(T^i)$ for some $j < i$, then we add wv to T^i, delete the first edge on the path of T^i from w to v and the number of edges in $T^i\langle x_0, N^{j'}(T^i)\rangle$ has been reduced for some $j' < i$. Secondly, if w is a predecessor of some member of $N^j(T^i)$ for some $j < i$, then we can again add vw to T^i and delete the first edge on the path from w to v in T^i. If w is not the parent of a vertex in B^j, then we contradict our choice of T^j as we have found a tree which meets the criteria for T^j with less edges. If w is the parent of some $y \in B^j$, then we also reach a contradiction unless $d_{T^i}(v) \geq \Delta - 2$. This is because we have produced a tree with $v \in N^j$, where $v \notin B^j$, and thus have reduced the size of B^j in T^j. However, we have shown that if $v \in C^i$, then $d_{T^i}(v) \geq \Delta - 2$.

Thus, for all $v \in N^i(T^i)$ with a G-neighbor in $T^i\langle x_0, N^i(T^i)\rangle$, $d_{T^i}(v) \geq \Delta - 2$. We can continue this process since $\Delta - 2 \geq \lceil \frac{n-1}{k} \rceil - 1 \geq 2$ (since we assumed that $\lceil \frac{n-1}{k} \rceil \geq 3$) and thus each time we get a vertex in B^i, it has at least one T^i-neighbor outside of $T^i\langle x_0, N^i(T^i)\rangle$.

Finally, note that this process ends when either there is an i such that $B^i = \emptyset$ or we run out of vertices of G, since for any $i \geq 2$, we saw in (1) that

$$|V(T^i\langle x_0, N^i(T^i)\rangle)| > |V(T^i\langle x_0, N^{i-1}(T^i)\rangle)|.$$

In the latter case, we reach a contradiction and thus must not have a counterexample, proving the theorem. In the former case, we let $T_0 = T^i$ for this i and $\mathbb{T} = T^i\langle x_0, N^i\rangle$. This completes the proof of the lemma. □

We begin by applying Lemma 1 to G to find $\mathbb{T}$, T_0 and $\mathcal{L}$. Note that for all $\ell \in \mathcal{L}$, $|N_G(\ell) \cap \mathbb{T}| = 1$ and since $\ell \in A_0$, we have that each $\ell \in \mathcal{L}$ has at least $k-1$ neighbors in G outside of $\mathbb{T}$. Further, we have that $G[\mathcal{L}]$ does not have any edges. We will use one other fact about T_0 throughout the remainder of the proof, namely, that in the construction of $\mathbb{T}$, we minimized the number of edges in $T_0\langle x_0, N^i\rangle$ for each i.

We define $\widehat{N}(T_0)$, or simply $\widehat{N}$, to be the set $N_G(\mathcal{L}) \setminus \mathbb{T}$. The main idea of the proof is to try to associate each edge of G between $\mathcal{L}$ and $\widehat{N}$ with a vertex in $V(G) \setminus V(\mathbb{T})$. Given $A, B \subset V(G)$ with $A \cap B = \emptyset$, write $G[A, B]$ for the graph with vertex set $A \cup B$ and edge set $E(G[A, B]) = \{e \in E(G) : e = ab, a \in A, b \in B\}$. Thus, we would like to construct a map with domain the set of edges of $G[\mathcal{L}, \widehat{N}]$ and range $V(G) \setminus V(\mathbb{T})$. We construct this map to get a lower bound on $|V(G) \setminus V(\mathbb{T})|$. Ideally, this map would be an injection, for if this were the case, there would be at least $\Delta + 1$ vertices in $\mathbb{T}$ and $k-1$ vertices outside $\mathbb{T}$ for each vertex in $\mathcal{L}$. Thus, we would see:

$$\begin{aligned} n = V(G) &\geq |V(\mathbb{T})| + |V(G) \setminus V(\mathbb{T})| \\ &\geq 1 + \Delta + (k-1)\Delta \\ &\geq 1 + k\Delta \\ &\geq 1 + k\left(\left\lceil \frac{n-1}{k} \right\rceil + 1\right) \\ &\geq n + k, \end{aligned} \tag{2}$$

a contradiction. In fact, we shall show that while we cannot guarantee an injection from edges of $G[\mathcal{L}, \widehat{N}]$ to $V(G) \setminus V(\mathbb{T})$, we can get something close, i.e., a map which is injective except on a specific set of edges.

The map that we are aiming for is not hard to describe. For $v \in \widehat{N}$, let the *packet* of v be all of the edges of $G[\mathcal{L}, \widehat{N}]$ incident to v. We will then map edges in a packet of $v \in \widehat{N}$ to a set of unique vertices of $V(G) \setminus V(\mathbb{T})$. This set of vertices will be associated with each $v \in \widehat{N}$ and, in fact, consist of vertices in the closed T_0-neighborhood of v.

We shall begin by partitioning the vertices of $\widehat{N}$. We do this according to whether the vertices are shared or unshared as neighbors of $\mathcal{L}$ as follows. Let

$$U = \{u \in \widehat{N} : \exists! \ell \in \mathcal{L} \text{ such that } u\ell \in E(G)\}$$

and

$$S = \{s \in \widehat{N} : |N_G(s) \cap \mathcal{L}| \geq 2\}$$

(see Figure 2). Note that $S \neq \emptyset$ since otherwise we could simply map an edge in $G[\mathcal{L}, \widehat{N}]$ to the unique vertex in U to which it is incident. This gives an injection and thus (2) provides a contradiction. We would like to show that each vertex in S has large tree degree. We begin by showing that $S \subseteq V(T_0)$.

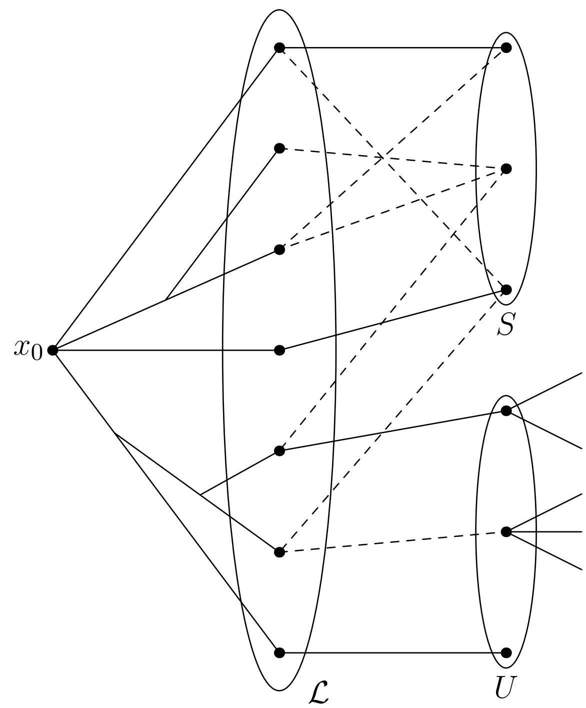

FIGURE 2. An example of U and S with $\Delta = 4$.

Claim 1. $S \subseteq V(T_0)$.

Proof. Recall that we let G be an edge-minimal counterexample. Suppose that $x \in S$, but $x \notin V(T_0)$. Since $x \in S$, there exist $u, v \in \mathcal{L}$ such that $ux, vx \in E(G)$. Note that $ux, vx \notin E(T_0)$ since $x \notin V(T_0)$. Further, note that since there are no edges of G between vertices in $\mathcal{L}$, $uv \notin E(G)$.

Construct a new graph G' from G by deleting both ux and vx from $E(G)$ and adding uv. Note that this operation may disconnect x from the rest of the graph, and if this is the case, we let $V(G') = V(G) \setminus x$. Then $|V(G')| \leq |V(G)| = n$. Further, note that $A_0 \subset V(G')$ because it is not possible for x to be in A_0 and not in T_0. By our construction, $\delta_{G'}(A_0) = k$. Also, $|E(G')| = |E(G)| - 1$, and so G' cannot be a counterexample to the theorem. Thus, there exists a tree T' in G' such that $V(T') \supset A_0$ and $\Delta_{T'}(A_0) \leq \lceil \frac{|V(G')|-1}{k} \rceil \leq \lceil \frac{n-1}{k} \rceil$. If T' does not contain the edge uv, then T' is a subgraph of G which satisfies the conditions of the theorem. If T' does contain uv, then we can remove uv from T' and add ux and vx to get a subtree $\bar{T}$ of G. The degrees of u and v have not changed in this process and the degree of x, while it has gone up by two, does not effect $\Delta_{\bar{T}}(A_0)$ since $x \notin A_0$. In either case, we have contradicted our choice of G as a counterexample. $\square$

We will now begin our series of claims which will help in our count of vertices in $V(G) \setminus V(\mathbb{T})$. To each vertex in S, we will attempt to associate unique vertices in $V(G) \setminus V(\mathbb{T})$. We begin by showing that every vertex of S has large tree degree.

Claim 2. *For any* $s \in S$, $d_{T_0}(s) \geq \Delta - 1 \geq \lceil \frac{n-1}{k} \rceil$.

Proof. Let $s \in S$. We know by Claim 1 that $s \in T_0$. Suppose that the claim is false, i.e., $d_{T_0}(s) < \Delta - 1$. Let ℓ be the unique predecessor of s in $\mathcal{L}$. Since $s \in S$, there is another vertex in $\mathcal{L}$, say v, such that $vs \in E(G)$. Note that this edge cannot be in T_0, or there would be a cycle in T_0. But, then, we add vs to T_0 and delete the first edge on the path in T_0 from v to x_0 and we have decreased the number of edges in some $T_0\langle x_0, N^i\rangle$, a contradiction to the way T_0 was chosen. We also note that this process does not create new vertices of degree Δ since $d_{T_0}(s) < \Delta - 1$. Thus, the claim is proved. □

We would like to show that to each $s \in S$, we can associate $\Delta - 1$ unique vertices (not in U). If this were possible, these vertices would be the image of the packet of s under our map. In order to do this, we need a claim that gives the structure of $T_0[\widehat{N}]$.

Claim 3. *For any $s \in S$, s cannot be the predecessor in T_0 of any $x \in \widehat{N}$.*

Proof. Suppose not so that $x \in \widehat{N}$ is an ancestor of $s \in S$. We note that x cannot be adjacent in T_0 to any $\ell \in \mathcal{L}$ since s is a predecessor of x. Since $x \in \widehat{N}$, x is adjacent in G to some $v \in \mathcal{L}$. We claim that by adding vx to T_0 and deleting sx, we reach a contradiction to our choice of T_0. To that end, let T' be the resulting tree. Note that G' does not necessarily meet the requirements of Lemma 1. However, T' is still in $\mathcal{T}_{x_0}^{\min}$ since $d_{T_0}(v) < \Delta - 1$, and so in adding vx we do not create a vertex of degree Δ. Also, we have decreased the degree of s. If $d_{T'}(s) = \Delta - 1$, we have contradicted the minimization of the number of vertices of degree Δ. On the other hand, if $d_{T'}(s) = \Delta - 2$, then we may argue similarly to the proof of Claim 2 and reach a contradiction to the construction of $\mathbb{T}$. Namely, since $s \in S$, it is adjacent in $G - T_0$ to some $y \in \mathcal{L}$ that is not its predecessor. By adding ys to T' and deleting the first edge on the path from y to x_0 in T', we have contradicted the minimality of $T_0\langle x_0, N^i\rangle$ for some i. In either case, we have reached a contradiction. Thus, s cannot in fact be the predecessor of x and the claim is proved. □

Note that Claim 3 implies that two closed T_0-neighborhoods of vertices in S can intersect in at most one vertex. This allows us to find our desired partition of vertices in $V(G) \setminus V(\mathbb{T})$. We shall now show that we can make this partition so that each vertex $s \in S$ has at least $\Delta - 1$ vertices in $V(G) \setminus (V(\mathbb{T}) \cup U)$ uniquely associated with it. Of course, in doing this, we must be careful to avoid vertices of U.

Claim 4. *There are at least $(\Delta - 1)|S|$ vertices in $V(G) \setminus (V(\mathbb{T}) \cup U)$.*

Proof. We begin by noting that the closed T_0-neighborhood of a vertex v, denoted $\bar{N}_{T_0}(v)$, is simply $\{v\} \cup N_{T_0}(v)$. We will show that a subset of $\cup_{v\in S}\bar{N}_{T_0}(v)$ will do for the Claim. Firstly, note that the subgraph of T_0 induced by $\cup_{v\in S}\bar{N}_{T_0}(v)$ is a forest. It may not be connected as two closed T_0-neighborhoods may be connected through other parts of T_0. Let q be the number of components of this forest. Then, we note that the number of vertices of this forest is q larger than the number of edges. However, each vertex of S has T_0-degree of at least $\Delta - 1$, by Claim 2, and further that the number of edges in the forest must be at least $(\Delta - 1)|S|$ since none of these edges are between members of S, by Claim 3. Thus, we see that the number of vertices of this forest must be at least $(\Delta - 1)|S| + q$. However, for each component of this forest, we may lose one vertex as it may already be counted. Note that the first vertex in the path in T_0 from x_0 in each component of our forest may be in either U, and thus already associated with another vertex in $\widehat{N}$, in $\mathcal{L}$ and thus already accounted for in our count or in another closed T_0 neighborhood of a predecessor in S. However, we then see that we still have $(\Delta - 1)|S|$ vertices in $V(G) \setminus (V(\mathbb{T}) \cup U)$, completing the proof of the claim. □

Now we must bound the size of S. Since by Claim 4, for each vertex of S we get at least $\Delta - 1 \geq \lceil (n-1)/k \rceil$ unique vertices, we find that

$$\begin{aligned} n &\geq 1 + \Delta + |S|(\Delta - 1) \\ &\geq 1 + \left\lceil \frac{n-1}{k} \right\rceil + 1 + |S| \left\lceil \frac{n-1}{k} \right\rceil. \end{aligned}$$

Consequently,

$$\begin{aligned}
|S| &\leq \frac{n - \lceil (n-1)/k \rceil - 2}{\lceil (n-1)/k \rceil} \\
&\leq \frac{nk - (n-1) - 2k}{n-1} \\
&= \left(\frac{n-2}{n-1}\right) k - 1 \\
&\leq k - 1. \qquad (3)
\end{aligned}$$

We know that each vertex of S is associated with $\Delta - 1 \geq \lceil \frac{n-1}{k} \rceil$ unique vertices by Claim 4. However, each vertex of S could be adjacent to all of $\mathcal{L}$ and thus incident to $|\mathcal{L}| \geq \Delta$ edges in $G[\mathcal{L}, \widehat{N}]$. Let $|\mathcal{L}| = \lceil \frac{n-1}{k} \rceil + l$ and note that $l \geq 1$ since $|\mathcal{L}| \geq \Delta \geq \lceil \frac{n-1}{k} \rceil + 1$. For each $s \in S$, the size of the image of the packet of s could be l less than the size of the packet, since we can only guarantee $\Delta - 1$ unique vertices associated with each $s \in S$. However, each vertex in $\mathcal{L}$ has $k-1$ edges to $\widehat{N}$, giving us the factor we need. Thus, we see that $|V(G) \setminus V(\mathbb{T})| \geq (k-1)|\mathcal{L}| - |S|l$. Using this and the fact that $|S| \leq k-1$, we see, in fact,

$$\begin{aligned}
n &\geq |V(\mathbb{T})| + |V(G) \setminus V(\mathbb{T})| \\
&\geq 1 + |\mathcal{L}| + (k-1)|\mathcal{L}| - |S|l \\
&= 1 + k|\mathcal{L}| - |S|l \\
&= 1 + k\left(\left\lceil \frac{n-1}{k} \right\rceil + l\right) - (k-1)l \\
&\geq 1 + n - 1 + kl - kl + l \\
&= n + l,
\end{aligned}$$

which, since $l \geq 1$, gives the desired contradiction and completes the proof.

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