Three Problems in Rationing Capital

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I. *Introduction*

Corporate executives face three tasks in achieving good financial management. The first is largely administrative and consists in finding an efficient procedure for preparing and reviewing capital budgets, for delegating authority and fixing responsibility for expenditures, and for finding some means for ultimate evaluation of completed investments. The second task is to forecast correctly the cash flows that can be expected to result from specified investment proposals, as well as the liquid resources that will be available for investment. The third task is to ration available capital or liquid resources among competing investment opportunities. This article is concerned with only this last task; it discusses three problems in the rationing of capital, in the sense of liquid resources.

1. Given a firm's cost of capital and a management policy of using this cost to identify acceptable investment proposals, which group of "independent" investment proposals should the firm accept? In other words, how should the firm's cost of capital be used to distinguish between acceptable and unacceptable investments? This is a problem that is typically faced by top management whenever it reviews and approves a capital budget.

Before presenting the second problem with which this paper deals, the use of the word "independent" in the preceding paragraph should be explained. Investment proposals are termed "independent" - although not completely accurately - when the worth of the individual investment proposal is not profoundly affected by the acceptance of others. For example, a proposal to invest in materials-handling equipment at location A may not profoundly affect the value of a proposal to build a new warehouse in location B. It is clear that the independence is never complete, but the degree of independence is markedly greater than for sets of so-called "mutually exclusive" investment proposals. Acceptance of one proposal in such a set renders all others in the same set clearly unacceptable-or even unthinkable. An example of mutually exclusive proposals would be alternative makes of automotive equipment for the same fleet or alternative warehouse designs for the same site. The choice among mutually exclusive proposals is usually faced later in the process of financial management than is the initial approval of a capital budget. That is, the decision as to which make automotive equipment to purchase, for example, typically comes later than the decision to purchase some make of equipment

2. Given a fixed sum of money to be used for capital investment, what group of investment proposals should be undertaken? If a firm pursues a policy of fixing the size of its capital budget in dollars, without explicit cognizance of, or reference to, its cost of capital, how can it best allocate that sum among competing investment proposals? This problem will be considered both for proposals which require net outlays in only one accounting period and for those which require outlays of more than one accounting period. In the latter case, special difficulties arise.

3. How should a firm select the best among mutually exclusive alternatives? That is, when the management of an enterprise, in attempting to make

> concrete and explicit proposals for expenditures of a type, which is included in an approved capital budget, develops more than one plausible way of investing money in conformance with the budget, how can it select the "best" way?

After presenting our solutions to these three problems, we shall discuss the solutions implied by the rate-of-return method of capital budgeting.[1] These solutions are worthy of special attention, since they are based on a different principle from the solutions that we propose and since the rate-of-return method is the most defensible method heretofore proposed in the business literature for maximizing corporate profits and net worth.

II. *The Three Problems*

A. GIVEN THE COST OF CAPITAL, WHAT GROUP OF INVESTMENTS SHOULD BE SELECTED?

The question of determining the cost of capital is difficult, and we, happily, shall not discuss it. Although there may be disagreement about methods of calculating a firm's cost of capital, there is substantial agreement that the cost of capital is the rate at which a firm should discount future cash flows in order to determine their present value.[2] The first problem is to determine how selection should be made among "independent" investment proposals, given this cost or rate.

Assume that the firm's objective is to maximize the value of its net worth-not necessarily as measured by the accountant but rather as measured by the present value of its expected cash flows. This assumption is commonly made by economists and even business practitioners who have spoken on the subject. It is equivalent to asserting that the corporate management's objective is to maximize the value of the owner's equity or, alternatively, the value of the owner's income from the business. Given this objective and agreement about the significance of the firm's cost of capital, the problem of selecting investment proposals becomes trivial in those situations where there is a well-defined cost of capital; namely, proposals should be selected that have positive present values when discounted at the firm's cost of capital. The things to discount are the net cash flows resulting from the investments, and these cash flows should take taxes into account.

There is nothing unusual or original about this proposed solution. It is identical with that proposed by Lutz and Lutz[3] and is an economic commonplace. Joel Dean in his writings has developed and recommended a method, which typically yields the same results for this problem, although the principle of solution is somewhat different, as is discussed later in this article.

The principle of accepting all proposals having positive present value at the firm's cost of capital is obvious, since the failure to do so would clearly mean foregoing an available increment in the present value of the firm's net worth. The principle is discussed here only because it seems a useful introduction to the somewhat more complicated problems that follow. An interesting property of this principle is that adherence to it will result in the present value of the firm's net worth being at a maximum at all points in time.

B. GIVEN A FIXED SUM FOR CAPITAL INVESTMENT, WHAT GROUP OF INVESTMENT PROPOSALS SHOULD BE UNDERTAKEN?

Some business firms-perhaps most-do not use the firm's cost of capital to distinguish between acceptable and unacceptable investments but, instead, determine the magnitude of their capital budget in some other way that results in fixing an absolute dollar limit on capital expenditures. Perhaps, for example, a corporate management may determine for any one year that the capital budget shall not exceed estimated income after taxes plus depreciation allowances, after specified dividend payments. It is probable that the sum fixed as the limit is not radically different from the sum that would be expended if correct and explicit use were made of the firm's cost of capital, since most business firms presumably do not long persist in policies antithetical to the objective of making money. (The profit-maximizing principle is the one that makes use of the firm's cost of capital, as described previously.) Nevertheless, there are probably some differences in the amount that would be invested by a firm if it made correct use of the firm's cost of capital and the amount that would be invested if it fixed its capital budget by other means, expressing the constraint on expenditures as being a maximum outlay. At the very least, the differences in the ways of thinking suggest the usefulness to some firms of a principle that indicates the "best" group of investments that can be made with a fixed sum of money.

The problem is trivial when there are net outlays in only one accounting period-typically, one year. In such cases, investment proposals should be ranked according to their present value-at the firm's cost of capital-per dollar of outlay required. Once investment proposals have been ranked according to this criterion, it is easy to select the best group by starting with the investment proposal having the highest present value per dollar of outlay and proceeding down the list until the fixed sum is exhausted.[4]

The problem can become more difficult when discontinuities are taken into account. For large firms, the vast majority of investment proposals constitute such a small proportion of their total capital budget that the problems created by discontinuities can be disregarded at only insignificant cost, especially when the imprecision of the estimates of incomes is taken into account. When a project constitutes a large proportion of the capital budget, the problem of discontinuities may become serious, though not necessarily difficult to deal with. This problem can become serious because of the obvious fact that accepting the large proposal because it is "richer" than smaller proposals may preclude the possibility of accepting two or more smaller and less rich proposals, which, in combination, have a greater value than the larger proposal. For example, suppose that the total amount available for investment were $1,000 and that only three investment proposals had been made: one requiring a net outlay of $600 and creating an increment in present value of $1,000 and two others, each requiring a net outlay of $500 and each creating an increment in present value of $600. Under these circumstances, the adoption of the richest alternative, the first, would mean foregoing the other two alternatives, even though in combination they would create an increment in present value of $1,200 as compared with the increment of $1,000 resulting -from the adoption of the richest investment alternative. Such discontinuities deserve special attention, but the general principles dealing with them will not be worked out here, primarily because we do not know them.

We shall, however, deal with the more serious difficulties created by the necessity to choose among investment proposals some of which require net cash outlays in more than one accounting period. In such cases a constraint is imposed not only by the fixed sum available for capital investment in the first period but also by the fixed sums available to carry out present commitments in subsequent time periods. Each such investment requires, so to speak, the use of two or more kinds of money-money from the first period and money from each subsequent period in which net outlays are required. We shall discuss only the case of investments requiring net outlays in two periods, for simplicity of exposition and because the principle although not the mechanics - is the same as for investments requiring net outlays in more than two periods.

Let us start with a very simple case. Suppose that all the available opportunities for investment that yield a positive income can be adopted without exceeding the maximum permitted outlay in either time period one or time period two. Clearly, no better solution can be found, because all desirable opportunities have been exhausted. This simple case is mentioned not because of its practical importance, which is admittedly slight, but because it may clarify the more complicated cases that follow.

Next, consider a slightly more complicated case. Suppose that the opportunities available require more funds from either time period one or two than are permitted by the imposed constraints. Under these circumstances the problem becomes somewhat more complicated, but it still may not be very complicated. It is still relatively simple if (a) the best -use of money available in period one does not exhaust the money available in period two or (b) the best use of money available in period two does not exhaust the money available in period one. In either case the optimum solution-that is, the solution which results in the greatest increment in the net worth of the firm, subject to the two stated constraints-is the one that makes the best possible use of the funds available for investment in one of the two time periods.

This statement is justified by the following reasoning. The imposition of additional restrictions upon the freedom of action of any agency can obviously never increase the value of the best opportunity available to that agency. In the problem at hand, this means that the imposition of an absolute dollar constraint or restriction in time period two can never make it possible to make better use of dollars available in time period one than would have been possible in the absence of that constraint. Thus, if the best possible use is made of the dollars available in time period one, the imposition of a restriction relating to time period two can never mean increased possibilities of profit from the use of funds available in time period one. Therefore, the maximization of the productivity of dollars available in time period one will constitute a maximization of productivity subject to the two constraints as well as to the one constraint. The reasoning is equally valid if we start with the constraint referring to time period two and maximize productivity of money available in that time period and then think of the effect of the additional constraint imposed for time period one.

Unfortunately, typical circumstances will probably make the relatively simple solutions unavailable. The solution to the relatively complex problem will-abstracting from discontinuities-require expending the full amount available for investment in each period. To illustrate how the solution is to be reached, consider the average actual net outlay of the two periods as being an outlay in a single "virtual" period and consider the average net outlay that is permitted by the constraints as being the average permitted outlay for the "virtual" period. Plan a budget for this "virtual" period according to the method of the one-period problem with which this section begins. That is, ration the capital available in the "virtual" period among the available investment opportunities so as to maximize the firm's net worth according to the principles stated in the discussion of the one period problem. If, by accident, this budget happens to require precisely those outlays, which are permitted for the first and second periods, it is easy to see that the problem has been solved. No other budget with a higher present value can be devised within the stated constraints for periods one and two.

Typically, the happy accident referred to in the preceding paragraph will not occur. The optimum use of the average amount available for investment in the two periods will typically result in expending too much in one period and not so much as is permitted in the other. Indeed, the happy accident was mentioned only as a step in explaining one method that will work. Though a simple average will almost never work, there is always some weighted average that will, and it can be found by trial and error. We shall describe in some detail a method that is mathematically equivalent to this method of weighted averages. In this method the solution is found by choosing, for suitable positive constants p_1 and p_2, those, and only those, proposals for which the following quantity is positive: $y - p_1c_1 - p_2c_2$. Here y is the present value of the proposal; c_1 and c_2 are the present values of the net outlays required in the first and second periods, respectively; and the multipliers p_1 and p_2 are auxiliary quantities for which there does not seem to be aid immediate interpretation but that nonetheless help in solving the problem.[5]

Initially, the values of p_1 and p_2 will be determined by judgment. Subsequently, they will be altered by trial and error until the amounts to be expended in the first and second period, according to the rule just enunciated, are precisely the amounts permitted by the constraints. The initial choice of values for ¢1 and ¢2 is not very important, since a graphical process can usually lead rapidly to the correct values.

Table 1

Investment	*Outlay – Period 1 (c_1)*	*Outlay – Period 2 (c_2)*	*Present Value of Investment*
A..........................	*$12*	*$ 3*	*$14*
B..........................	*54*	*7*	*17*
C..........................	*6*	*6*	*17*
D..........................	*6*	*2*	*15*
E..........................	*30*	*35*	*40*
F..........................	*6*	*6*	*12*
G..........................	*48*	*4*	*14*
H..........................	*36*	*3*	*10*
I..........................	*18*	*3*	*12*

Certain special possibilities are worth noting. Proposals of positive present value may have negative cost, that is, release cash, for either period. Some proposals of zero or negative present value may be acceptable because they release cash for one period or both. All such possibilities are automatically covered by the rule as stated and by the rules to be given for later problems.

Finding the correct values for p_1 and p_2 is sometimes not easy - especially when combined with the problem of selecting among mutually exclusive alternatives - but the task is usually as nothing compared to the interests involved or compared to many everyday engineering problems.[6] The following example may clarify the process.

Nine investments have been proposed. The present value of the net outlays required in the first and second time periods and the present values of the investments are as shown in Table 1. The finance committee has stated that $50 and $22 will be available for capital investment in the first and second periods, respectively. We shall consider these amounts to have present values of $50 and $20, respectively. According to the principle stated above, we must now find appropriate multipliers, p_1 and p_2.

Multipliers p_1 and p_2 were initially set at 1 and 3, respectively. With these values, only for investment *d* was the expression $(y - p_1c_1 - p_2c_2)$ positive and therefore acceptable. This would have resulted in net outlays of only $6 and $2 in periods one and two, respectively. Clearly, the values initially chosen for p_1 and p_2 were too great. On the other hand, values of 0.1 and 0.5 for p_1 and p_2 respectively, are too low, resulting in a positive value of $(y - p_1c_1 - p_2c_2)$ for all investments and required outlays in periods one and two far exceeding the permitted outlays.

Values of 0.33 and 1 for p_1 and p_2 result in a near-perfect fit. The expression $(y - p_1c_1 - p_2c_2)$ is positive for investments a, c, d, f, and i. These investments require outlays of $48 and $20 in the first and second periods, as near the permitted outlays of $50 and $20 as discontinuities permit. No other group of investments that is possible within the stated constraints has a greater present value than $70, the present value of this group.[7]

C. SELECTING THE BEST AMONG MUTUALLY EXCLUSIVE ALTERNATIVES

Before moneys are actually expended in fulfillment of an approved capital budget, the firm usually considers mutually exclusive alternative ways of making the generally described capital investment. When the firm is operating without an absolute limit on the dollars to be invested, the solution to the problem of selecting the best alternative is obvious. (Throughout this article, it is assumed that decisions regarding individual investment proposals do not significantly affect the firm's cost of capital.) The best alternative is the one with the greatest present value at the firm's cost of capital.

When the firm is operating subject to the constraint of an absolute dollar limit on capital expenditures, the problem is more difficult. Consider, first, the case in which there are net outlays in only one time period. The solution is found by the following process:

1. From each set of mutually exclusive alternatives, select that alternative for which the following quantity is a maximum: $y - pc$. Here y is the present value of the alternative; c is the net outlay required; and p is a constant of a magnitude chosen initially according to the judgment of the analyst. (Remember that the alternative of making no investment - that is accepting $y = 0$ and $c = 0$ - is always available, so that the maximum in question is never negative.)

2. Compute the total outlays required to adopt all the investment proposals selected according to the principle just specified.

4. If the total outlay required exceeds the total amount available, p should be increased; if the total amount required is less than the amount available for investment, p should be reduced. By trial and error, a value for p can be found that will equate the amount required for investment with that available for investment.

It should be clear that, as the value of p is increased, the importance of the product; pc, increases, with a consequent increase in the probability that in each set of mutually exclusive alternatives, an alternative will be selected that requires a smaller net outlay than is required with a smaller value for p. Thus increasing p tends to reduce the total amount required to adopt the investment proposals selected according to the principle indicated in (1) above. Conversely, reducing p tends to increase the outlay required to adopt the investment proposals selected according to this principle.

When there are net outlays in more than one period, the principle of solution is the same. Instead of maximizing the quantity $(y - pc)$, it is necessary to maximize the quantity $(y - p_1c_1 - p_2c_2)$, where again c_1 and c_2 are the net outlays in the first and second periods and p_1 and p_2 are auxiliary multipliers.

Up to this point, we have not discussed the problem of rationing capital among both independent investment proposals and sets of mutually exclusive investment proposals. Superficially, this problem seems different from the one of rationing among mutually exclusive proposals only, but in fact the problems are the same. The identity rests upon the fact that each so-called "independent" proposal is and should be considered a member of the set of proposals consisting of the independent proposal and of the always-present proposal to do nothing. When independent proposals are viewed in this way, it can be seen that the case of rationing simultaneously among independent proposals and sets of mutually exclusive proposals is really just a special case of rationing among mutually exclusive proposals according to the principles outlined in the preceding paragraph.

The mechanics of solution are easily worked out. All that is required in order to make the solution the same as the solution for what we have called "mutually exclusive" sets of alternatives is that each so-called "independent" proposal be treated as a member of a mutually exclusive set consisting of itself and of the alternative of doing nothing. Once this is done, it is possible to go into the familiar routine of selecting from each set that proposal for which the expression $(y - pc)$, or its appropriate modification to take account of constraints existing in more than one time period, is a maximum. Again, of course, that value of p will have to be found which results in matching as nearly as discontinuities permit the outlays required by the accepted proposals with the outlays permitted by the stated budgetary constraints.

III. *Some Comparisons with the Rate-of-Return Method of Capital Rationing*[8]

Since the rate-of-return method of capital rationing is fully described elsewhere, we shall describe it only briefly.[9] As in the methods described previously, attention is focused exclusively on net cash flows rather than on the data produced by conventional accounting practices. Investment proposals are ranked according to their "rate of return," defined as that rate of discounting which reduces a stream of cash flows to zero, and selected from this ranking, starting with the highest rate of return.

The rate-of-return solution to the three problems that are the subject of this paper is discussed below.

A. GIVEN THE COSTS OF CAPITALS WHAT GROUP OF INVESTMENTS SHOULD BE SELECTED?

The rate-of-return solution to the problem of selecting a group of independent proposals, given the firm's cost of capital, is to accept all investment proposals having a rate of return greater than the firm's cost of capital. This solution is necessarily identical with the solution proposed previously, except when the present value of some of the proposals is other than a steadily decreasing function of the cost of capital. An intuitive substantiation of this statement is achieved by an understanding of Figure 1.

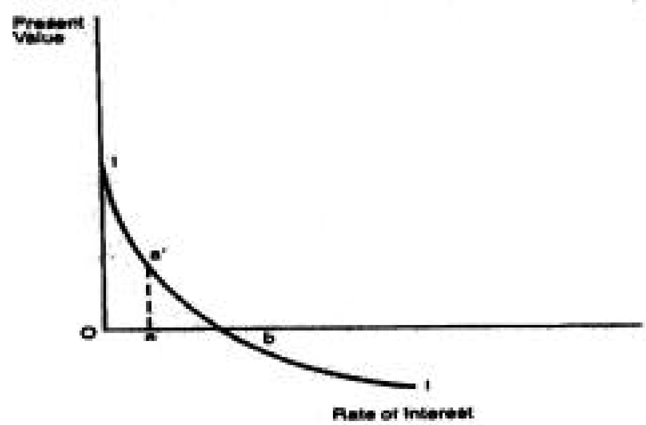

FIGURE 1

In Figure 1, *I-I* indicates the present value of an investment at different rates of interest; *Oa* is the firm's cost of capital; *Ob* is the rate of return on the investment; and *aa'* is the present value of the investment at the firm's cost of capital. It should be clear from the diagram that any proposal that has a positive ordinate (present value) at the firm's cost of capital will also have a rate of return (x-intercept) greater than the cost of capital. (However, it usually takes a little longer to find an intercept than to determine the value of an ordinate at one point.)

Under what circumstances can the present value of an investment proposal be something other than a steadily decreasing function of the cost of capital? Some investment proposals can intersect the *x*-axis at more than one point. In particular, investment proposals having initial cash outlays, subsequent net cash inflows, and final net cash outlays can intersect the x-axis more than once and have, therefore, more than one rate of return. Investments of this nature are rare, but they do occur, especially in the extractive industries. For example, an investment proposal might consist of an investment in an oil pump that gets a fixed quantity of oil out of the ground more rapidly than the pump currently in use. Making this investment would require an initial net outlay (for the new pump), subsequent net incremental cash inflow (larger oil production), and final net incremental cash outlay (the absence of oil production, because of its earlier exhaustion with the use of the higher-capacity new pump).[10] The present value of an investment in such a pump could look like Figure 2. In Figure 2, *I-I* indicates the present value of the investment; *Oa* is the firm's cost of capital; *Ob* and *Oc* are the two rates of return on the investment; and *aa'* is the present value of the investment at the firm's cost of capital.

The reasoning behind this apparent paradox of the double rate of return is as follows:

a) As the cost of capital of the firm approaches zero, the present value of the investment proposal approaches the algebraic sum of net cash flow and will be negative if this sum is negative.

b) As the cost of capital increases, the present value of the final net cash out-flow diminishes in importance relative to the earlier flows, and this diminution can cause the present value of the entire proposal to become positive.

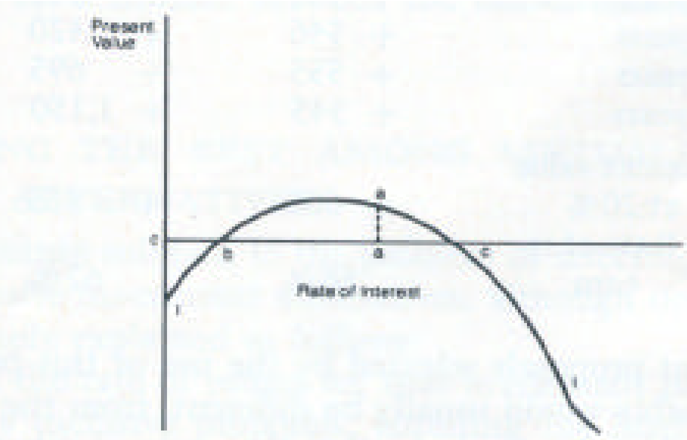

FIGURE 2

c) If the cost of capital continues to increase, the significance of all future cash flows tends to diminish, causing the present value of the proposal to approach the initial outlay as a limit.

The rate-of-return criterion for judging the acceptability of investment proposals, as it has been presented in published works, is then ambiguous or anomalous. This is in contrast to the clarity and uniform accuracy of the decisions indicated by the principle proposed earlier, which relates to the present value of an investment at the host of capital rather than to a comparison between the cost of capital and the rate of return.[11]

B. GIVEN A FIXED SUM FOR CAPITAL INVESTMENT, WHAT GROUP OF INVESTMENT PROPOSALS SHOULD BE UNDERTAKEN?

The rate-of-return solution to the problem of allocating a fixed sum of money-without reference to cost of capital-among competing proposals is to order the proposals according to their rate of return and to proceed down the ladder thus created until the available funds are exhausted. The group of investment proposals selected by the use of this principle can be

Table 2

Period	*Net Cash Flows* *Investment A*	*Investment B*
0- year………………	*- $ 85*	*- $ 90*
0-1 year ………………	*+ 17*	*+ 21*
1-2 years………………	*+ 35*	*+ 33*
2-3 years………………	*+ 68*	*+ 57*
3-4 years………………	*+ 131*	*+ 94*
4-5 years………………	*+ 216*	*+ 155*
5-6 years………………	*+ 357*	*+ 255*
6-7 years………………	*+ 546*	*+ 420*
7-8 years………………	*+ 555*	*+ 695*
8-9 years………………	*+ 345*	*+ 1,150*
Present Value at 20%.	*+ 606*	*+ 853*
Rate of Return…………	*66%*	*62%*

different, and probably would usually be different, from the group selected when the criterion is present value rather than rate of return. A difference between the two groups would not exist if the available capital funds were just equal to that amount which would permit investment in all those proposals having a rate of return at least equal to the firm's cost of capital and only those proposals, and if the anomalies mentioned under Section A were not present.

The preceding statements are equivalent to saying that the groups of investments that would be chosen by the use of the two principles or criteria would necessarily be the same only if the fixed sum to be invested happened to be the optimum sum and that investment of any other sum could result in selection of different groups of proposals by use of the two principles. This difference would result from the fact that the different principles can result in a different ranking of proposals within the group that would be accepted if the optimum amount were invested. Table 2 indicates the validity of the statement that the ordering of two investment proposals according to their rate of return can be contrary to their ordering according to their present value per dollar of outlay.

The example of Table 2 illustrates that a proposal with a higher rate of return can have a lower present value and that, therefore, the two rules can conflict. The present-value rule maximizes the present value of the firm's net worth-by definition-and the rate-of-return rule therefore may not.

This discrepancy is undoubtedly of small practical significance. In the first place, firms that ration their capital rationally use the firm's cost of capital as the constraint rather than an absolute dollar sum, and under such rational behavior the two rules yield the same results, with the exception noted previously. (Undoubtedly, no firms long persist in setting absolute dollar constraints that differ significantly in their effects from the cost of capital constraint.) In the second place, the present values of investment proposals, expressed as functions of the cost of capital, are often thoughtful enough not to intersect above the x-axis (the rate-of-interest axis), a necessary condition for a conflict between the rate-of-return and present-value principles.

C. SELECTING THE BEST AMONG MUTUALLY EXCLUSIVE ALTERNATIVES

The rate-of-return solution to the problem of selecting the "best" among mutually exclusive investment alternatives, although occasionally tricky in practice, is simply explained as follows:

1. Compute the rate of return for that investment proposal, among the set of mutually exclusive proposals, requiring the least initial net outlay.

2. If the rate of return on the investment requiring the smallest outlay exceeds the firm's cost of capital (or other cutoff rate), tentatively accept that investment. Next compute the rate of return on the incremental outlay needed for the investment requiring the second lowest outlay. If that rate exceeds the firm's cutoff rate, accept the investment requiring the greater outlay in preference to that requiring the lesser. Proceed by such paired comparisons (based on rates of return on incremental outlay) to eliminate all but one investment.

2. If the rate of return on the proposal requiring the least outlay does not exceed the firm's cutoff rate, drop it from further consideration, and compute the rate of return for the proposal requiring the next least outlay. If that rate exceeds the firm's cutoff rate, that investment proposal becomes the benchmark for the first paired comparison. If that rate does not exceed the firm's cutoff rate, drop that proposal from further consideration. The process just described is to be repeated until either a proposal is found with a rate of return exceeding the cost of capital or until all proposals have been eliminated because their rates of return do not exceed the cutoff rate.

The rate-of-return solution to the problem of selecting the best among mutually exclusive investment alternatives is especially subject to the ambiguities and anomalies mentioned under Section A, because the costs and revenues associated with incremental investments required for proposals included in mutually exclusive sets are much more likely to have unusual time shapes and reversals than are the costs and revenues associated with independent investments.

SUMMARY

We have given solutions to three problems in budgeting capital so as to maximize the net worth of the firm. The solutions that we have given differ in principle from those implied by the rate-of-return method of capital rationing. The difference in principle can lead to differences in behavior. Differences in behavior will be rare in coping with problems of the first and third sorts and will be relatively frequent for problems of the second sort. When differences do exist, the rate-of-return solution does not result in maximizing the present worth of the firm's net worth.

[1] This method was developed by Joel Dean, who has probably done more than anyone else in applying the formal apparatus of economics to the solution of capital budgeting problems in their business context.

[2] One of the difficulties with the concept of cost of capital is that in complicated circumstances there may be no one rate that plays this role. Still worse, the very concept of present value may be obscure.

[3] Friederich and Vera Lutz, *The Theory of Investment of the Firm* (Princeton: Princeton University Press, 1951.). The solution proposed here is identical with the maximization of V-C, where V is the present value of future inflows and C is the present value of future outflows. This is discussed in chap. ii of the Lutz book.

[4] We mention, for completeness, that the outlay or the present value or both for a proposal can be negative. Proposals for which the outlay alone is negative-something for nothing-are always desirable but almost never available. Proposals for which both the outlay and present value are negative can sometimes be acceptable if something sufficiently profitable can be done with ready cash expressed by the negative outlay. The rules which we shall develop can be extended to cover such cases.

[5] The multipliers, p_1 and p_2, are closely related to what are known in mathematics and in economics as “Lagrange multipliers”.

[6] It is true, however, that the numbers in engineering problems are less conjectural; hence the cost of calculation is more likely to be considered worthwhile.

[7] For the three-period problem, the relevant quantity is $(y - p_1c_1, - p_2c_2 - p_3c_3)$ rather than $(y - p_1c_1, - p_2c_2)$.

[8] Joel Dean has pioneered in the development of methods of capital rationing that have an understandable relationship to profit maximization, in contrast to methods still quite widely used in business that rely on such criteria as pay-back, average return on book investment, etc. The method that he advocates is called the "rate-of-return" method.

[9] See Joel Dean, Capital Budgeting (New York: Columbia University Press, 1951); "Measuring the Productivity of Capital," Harvard Business Review, January-February, 1954.

[10] These incremental flows are measured with reference to the flows that would have resulted from the use of the smaller pump. Thus the final net outlay is not absolute but rather by comparison with oil (money) that would have produced had the smaller pump been in use.

[11] The rate-of-return rule could be easily modified to remove this ambiguity or anomaly by specifying that the relevant rate of return is the one at which the investment is a decreasing function of the rate of interest.