LEIBNIZ, BERNOULLI AND THE LOGARITHMS OF NEGATIVE NUMBERS

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Abstract

In 1712, Gottfried Leibniz and John Bernoulli I engaged in a friendly correspondence concerning the logarithms of negative numbers. The publication of this correspondence in 1745 sparked an interest in the mathematical community on the topic. In this paper I will discuss the evolution of the logarithmic concept in the time leading up to their discussion. I will then give a synopsis of the correspondence followed by a description of a paper by Leonhard Euler in which he addresses the issue.

1 Evolution of the Logarithmic Concept

In order to understand the reasoning motivating the arguments made by Leibniz and Bernoulli during their discussion of the logarithms of negative numbers, we must first understand what the logarithm was at the time. Presently, in many math text books, one would not be surprised to find a chapter entitled, "The Exponential and Logarithmic Concepts." We take the deep connection between these two ideas for granted. Hence, one may be surprised to learn that this connection was not explicitly stated until the mid-eighteenth century, more than one hundred thirty years after the introduction of the logarithm.

The logarithm was invented independently by two men, John Napier (1550 - 1617) and Joost Bürgi (1552 - 1632), though the credit is usually given to the former, who published his work in 1614. By the time Bürgi published in 1620, Napier's work was already quite popular. John Napier, a Scotsman from Merchiston Castle, never held any academic positions, but rather performed mathematics as a hobby. Napier coined the term logarithm which, coming from Greek, means “number of ratios” or “reckoning number.” Napier published *Mirifici Logarithmorum Canonis Descriptio* in 1614. In this work, the Scotsman described his newly invented logarithm and included tables of logarithms. The posthumously published *Mirifici Logarithmorum Canonis Constructio*, of 1619, described his methods for calculating the logarithms.

Napier's logarithms worked as follows: he has two lines, one a line segment of finite length ($\overline{AB}$ in Figure 1) and the other, a ray of indefinite length. A point C is set in motion on $\overline{AB}$ with an initial speed equal to the length of $\overline{AB}$. Its subsequent speed is equal to $\overline{CB}$. At the same time that this point was set in motion, another point, C' is set off from A' with the same initial speed; however, this point maintains this same speed of the length of $\overline{AB}$.

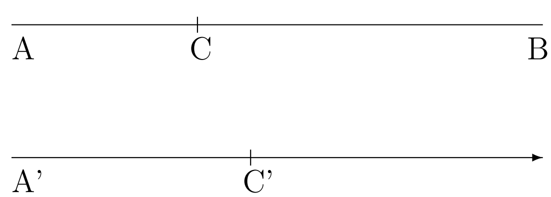

Figure 1. Napier's 'flowing points' concept of logarithm

Napier defines the logarithm of the length of $\overline{CB}$ to be the length of $\overline{A'B'}$. Napier's logarithm was originally invented in order to ease calculations performed by astronomers and navigators involving sines. The tables of sines present at the time existed to seven decimal places. Hence Napier chose 10^7 as the length of $\overline{A'B'}$. This yielded integral logarithms of sines. [7]

It can be shown that Napier's logarithm, which we will denote NapLog, can be expressed in terms of the modern natural logarithm as follows [6]:

$$NapLog(x) = 10^7 \log_e \left(\frac{10^7}{x} \right) \tag{1}$$

Now the raison d'être of the logarithm is the following set of properties:

$$\log(ab) = \log(a) + \log(b) \tag{2}$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b) \quad (3)$$

$$\log(a^n) = n\log(a) \quad (4)$$

NapLog has a property similar to (2), that is:

$$NapLog(ab) = NapLog(a) + NapLog(b) - NapLog(1) \quad (5)$$

In the summer of 1617, Henry Briggs, an English geometer who was impressed by the logarithm traveled to Scotland to visit its inventor. A famous account of their first encounter is as follows. When Briggs arrived, he was taken to Napier

> "where almost one quarter of an hour was spent, each beholding the other with admiration before one word was spoken. At last Mr. Briggs began: 'My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto Astronomy, viz. the Logarithms: but My Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy.' " [6]

Napier and Briggs discussed the logarithm and decided it would be best for it to have the property (2). The appendix to Napier's 1619 work, contains a revision of the logarithm to one which has this property.

At this point, logarithms were seen as a one to one correspondence between a geometric and an arithmetic progression. The geometric progression being represented by the point with changing velocity and its logarithm, the arithmetic progression represented by the point with constant velocity. Modern exponential notation had not yet been established and the exponential concept had not yet been developed, so there was no connection between logarithms and exponents at this point.

In the seventeenth century, the graphical representation of the logarithm in both rectangular and polar coordinates became popular. Descartes, Toricelli, Gregory and John Bernoulli all worked with the logarithmic curve in rectangular coordinates. In 1638, Descartes first described the logarithmic spiral in a letter to Mersenne. He did not though mention the logarithm at all in this letter. [1]

A Belgian mathematician named Gregory St. Vincent published, in multiple books, his *Opus Geometricum* (a work in which he claimed to have squared the circle). In one of the books of 1647, he says that if vertical lines are drawn from the hyperbola to the horizontal axis such that the areas of each of the quadrilaterals are equal, then the distance between the vertical lines form a geometric progression. St. Vincent in this description, like Descartes in his discussion of the spiral, does not mention logarithms at all. One of his students, Alfons Anton de Sarasa, was the first to state this fact using logarithms. Since logarithms were thought of as a one to one correspondence between arithmetic and geometric progressions, this was a fairly natural step to take. [1]

Another interesting result involving logarithms came in 1668 when Nicolaus Mercator published his booklet, *Logarithmotechnia*. One may recall that

$$\log(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \cdots \tag{6}$$

is often referred to as the Mercator series. In this booklet, Mercator divides out $1/(1+a)$ into the series $1-a+a^2-a^3+\ldots$ and then gives an description of what we know as $\int x^n dx = x^{n+1}/(n+1)$. He then integrates the terms of the series for $1/(1+a)$ and gives a few numerical values for the areas of the quadrilaterals formed under the hyperbola. He does not, however, explicitly state the formula (6). Some say that this is because many mathematicians of the time preferred specific results over general formulas. Wallis was the first to state Mercator's result in symbolic form. [1]

2 Controversy Over the Logarithms of Negative Numbers

Presently in mathematics, we use different domains all the time depending on the situation. We are careful to state exactly which sets we are working with for every problem. In the early eighteenth century, it was not so clear that this could be done. Mathematicians often had the view that either a concept worked for everything or for nothing at all.

The fact that certain concepts did not extend easily to negative and imaginary numbers is seen in the way that some referred to these numbers. Negative numbers were sometimes called false or defective numbers while imaginary numbers were sometimes called impossible numbers. [2]

The extension of the logarithm to negative and imaginary numbers would be a controversy which would last for more than a century. The German co-inventor of calculus, Gottfried Leibniz (1646 - 1716) and a member of the famous Swiss Bernoulli family, John Bernoulli I (1667 - 1748) would be amongst the first to discuss the topic. Even with the contemporary doubts about the validity of negative and imaginary numbers, Leibniz and Bernoulli each worked with them. In 1702, Bernoulli reached the result that

$$\arctan(x) = \frac{1}{2i}\log\left(\frac{1+ix}{1-ix}\right).$$

Letting x = 1, we have that

$$\log(i) = i\frac{\pi}{2}$$

While at the time, this result was seen merely as a novelty, we will see later that one man did not think so lowly of it.

Leibniz and Bernoulli were good friends who corresponded often about many topics. In Leibniz's controversy with Isaac Newton, concerning the invention of the calculus, Bernoulli would be one of Leibniz's most ardent supporters. However, they did not agree on everything as can be seen in a series of letters from March 16, 1712 to July 29, 1713. In this correspondence consisting of 12 letters back and forth, the two have a friendly debate concerning their beliefs on the logarithms of negative numbers.

The discussion started with a paper by Leibniz of 1712 in which he writes about a subject that mathematicians and philosophers of the time were writing about. The ratios 1:-1 and -1:1 were problematic. If the two ratios were the same, then the greater would be to the less the same as the less is to the greater. In this time, when one took a logarithm, it was thought to be the logarithm of the "measure of a ratio." So Leibniz states in this paper that a ratio can be considered imaginary or impossible if it has no logarithm. He says that $\log(-1/1) = \log(-1) - \log(1)$ and that $\log(-1)$ does not exist because then $\frac{1}{2}\log(-1) = \log(i)$ which he considered absurd. Notice that Leibniz uses the word imaginary to mean both non-existent and $\sqrt{-1}$. This double meaning will come into play later. Before the publication of this paper, Leibniz described this argument in a letter to Bernoulli which would be the first of their correspondence regarding this topic.

Bernoulli responded that since $\frac{dx}{x} = \frac{-dx}{-x}$, by integration we have that $\log(x) = \log(-x)$. So he claims that the logarithmic curve has two symmetric branches, similar to the hyperbola. Leibniz responds to this by saying that the differentiation rule for $log(x)$ does not apply to $\log(-x)$.

In his response he also mentions that $\log(-2)$ does not exist because then $\frac{1}{2}\log(-2) = \log(\sqrt{-2})$ which he again considers absurd.

Bernoulli responds to this claim by denying that $\frac{1}{2}\log(-2) = \log(\sqrt{-2})$ even though $\frac{1}{2}\log(2) = \log(\sqrt{2})$. He says that $\sqrt{2}$ is a mean proportional between 1 and 2 , but $\sqrt{-2}$ is not a mean proportional between -1 and -2. So he says that just as $\log(\sqrt{1 \times 2}) = \frac{1}{2}\log(2)$, we also have that $\log(\sqrt{-1 \times -2}) = \frac{1}{2}\log(-2)$.

Leibniz states his definition which is that "Logarithms are numbers in arithmetic progression, corresponding to numbers in geometric progression, of which one number may be 1 and another may be any positive number." So he says assume $\log(1) = 0$ and $\log(2) = 1$. He says that then that a negative number would never be a part of the geometric series described by these relations. He also argues that if $\log(\sqrt{2})$ existed, then $\frac{1}{2}\log(-2) = \log(\sqrt{2})$ because $\sqrt{-2}$ is the mean proportional between +1 and -2 and this would mean that $\log(\sqrt{-2}) = (\log(1) + \log(-2)) : 2 = \frac{1}{2}\log(-2)$.

Bernoulli responds to this by saying that he agrees that positive and negative numbers would not arise in the same geometric series but that negative numbers "determine their own peculiar series starting with -1, instead of +1"[2] and with this series, the logarithmic properties of negative numbers are the same as positive numbers. Bernoulli then proceeds to give a geometric argument, as he would often do. He draws the two branches of the hyperbola and then constructs two branches of the logarithmic curve by using proportional areas under the hyperbola. Bernoulli's argument involves passing across the asymptotes of the hyperbola, and uses the assumption that $\infty - \infty = 0$.

Leibniz responds that given $2^c = x$, if x = 1, then c = 0 and if x = 2, then c = 1. But if x = -1, then there is no c which satisfies the property. Bernoulli responds that these assumptions are arbitrary. He says that when c = 0, let x = -1. With this assumption he says we can obtain any negative x. In Leibniz's reply, he claims that his assumptions are more natural than Bernoulli's and he goes on to support this claim by pointing out contradictions that arise from Bernoulli's assumptions. If $2^0 = -1$ then 1, -1 and i all have the same logarithm. Also, unless 2^c is many valued, then 2^0 is equal to 1, -1, $\sqrt{-1}$, $\sqrt[4]{-1}$, $\sqrt[8]{-1}$ and so forth. He also points out that if $x^c = -2$ then $x^{2c} = +4$, but Bernoulli himself agreed that positive and negative numbers would not show up in the same geometric series.

In Bernoulli's reply he refutes all of Leibniz's contradictions. He once again claims that the half of a logarithm is not necessarily the logarithm of the square root by appealing to mean proportionals . He denies that 2^0 is equal to -1, 1, $\sqrt{-1}$, $\sqrt[4]{-1}$, $\sqrt[8]{-1}\ldots$ because his system only implies that 2^0 is equal to $\sqrt{-1 \times -1}$, $\sqrt{-1 \times -1 \times -1 \times -1}$ etc.

In the second to last letter of this correspondence, Leibniz says, "I have no time to disprove your objections to my doctrine which makes $\log(i)$ impossible, the double of impossibles impossible, $\log(n)$ the double of $\log(\sqrt{(n)})$. If you assume logarithms in which this is not so, that is nothing to me." In the final letter of July 29, 1713, Bernoulli says that Leibniz did not deny that his assumptions were arbitrary and so everything Bernoulli had been saying about $\log(-n)$ was true.

From this correspondence we can see that there was a great need for a new definition of logarithms. Merely referring to a correspondence between a geometric and an arithmetic series was not sufficient. Leibniz believed that +1 had to be a member of the series, whereas Bernoulli felt that -1 was just as valid. Leibniz seemed to value the properties (2, 3, 4 described on page 3 of this paper) of the logarithm whereas Bernoulli seemed more concerned with extending the logarithmic concept to negative numbers, even if it meant that these properties were no longer valid.

This correspondence was published in 1745, which is when the topic of logarithms of negative numbers began to gain interest in the mathematical community. While nothing was resolved by the debate between Leibniz and Bernoulli, it did manage to spark the curiosity of one of Bernoulli's students, Leonhard Euler.

3 Euler Addresses the Debate

Leonhard Euler (1707-1783), the prolific Swiss mathematician gained an interest in the topic of logarithms of negative numbers early in his life. When Euler was just 20 years old, he corresponded by letter with his teacher John Bernoulli on the subject. In this correspondence describes his arguments both for Bernoulli's position and against it. Bernoulli still held the same view that $\log(n) = \log(-n)$.

As mentioned earlier, the union of the logarithmic and exponential concepts had not yet taken place in a rigorous and widely accepted fashion. Often times, John Wallis is accredited with this connection. In his *Algebra* of 1685 he points out that in the two progressions crucial to the logarithm (arithmetic and geometric), the numbers in the arithmetic progression are really referring to exponents in the geometric series. Math historian Florian Cajori says, though, that "Wallis does not come out, resolutely, with the modern definition of a logarithm, and use it."[2]

In William Gardiner's *Tables of Logarithms* of 1742, he writes, "The common logarithm of a number is the Index of that power of 10 which is equal to the number." Only a few years after this, Euler would also define logarithms in terms of exponents in his *Introductio in Analysin* written in 1745 and published in 1748.

Between 1745 and 1749, Euler both corresponded with other mathematicians and wrote paper on the nature of logarithms and negative numbers. His main correspondence on the topic was with the Frenchman, Jean le Rond D'Alembert (1717-1783). D'Alembert held the view of John Bernoulli and Euler tried to convince him otherwise. Euler and D'Alembert would later have a falling out, but this correspondence took place when they were still on good terms with each other.

Euler wrote two articles concerning the logarithms of negative numbers. The first, *Sur les logarithmes* was written in 1747 but for some reason was not published until 1862. The second paper, *De la controverse entre Messrs. Leibniz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires*, which I will describe here, was written in 1749 and published in 1751.

In this paper of 1749, Euler begins by describing the position of Bernoulli. He retells some of Bernoulli's specific arguments from the correspondence. He then says that the strongest argument for Bernoulli's position is the following which requires the acceptance of one of the natural laws of logarithms; namely, that $\log(a^n) = n\log(a)$. Since $(-a)^2 = a^2$, then their logarithms are equal, so $\log(-a)^2 = \log(a^2)$, so then $2\log(-a) = 2\log(a)$ and hence $\log(-a) = \log(a)$, which is exactly the position of Bernoulli. Although this is a strong argument for the position of Bernoulli, it would seem that Bernoulli himself would not have made this case. In his letter of June 7, 1713 to Leibniz, Bernoulli says that "Twice $\log(-n)$ is not $\log n^2$." [2]

He then proceeds to give objections to the arguments for Bernoulli's position. Recall that in Bernoulli's first letter of the correspondence, he claims that since $\frac{dx}{x} = \frac{-dx}{-x}$, by integration we can conclude that $\log(x) = \log(-x)$. Leibniz refuted this by claiming that the differentiation rule for $\log(x)$ does not hold for $\log(-x)$. Euler first points out that Leibniz's argument was not supported by any valid reason and that if his claim were true, it would overturn all that was known about the differential properties of the logarithm.

He then says that a much better argument against this argument would be Bernoulli's disregard for the integration constant. Indeed, by ignoring the fact that the integrals mentioned above would differ by a constant, Bernoulli inherently assumed that $log(-1) = 0$, which is the logarithm at the heart of the debate.

Euler then moves on to the argument mentioned in the paragraph preceding the last. He says that if $\log(-1) = \log(1) = 0$, then $\log(i) = 0$ as well. But this contradicts Bernoulli's own finding of 1702, that $\log(i) = i\frac{\pi}{2}$. Euler was not willing to accept this. He then points out that if we accept that $\log(-1) = w$ for some $w \neq 0$, then $2\log(-1) = 2w$. But this implies that $\log(-1) = 2w \neq 0$, which no one would be willing to accept.

After this section of the article, Euler does the same thing for Leibniz's position: he gives arguments for it and then gives objections to these arguments. The following, one of his 'proofs' that $\log(-1) \neq 0$, is also a great demonstration of his liberal usage of infinite series. He observes that

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots.$$

Setting $x = -2$, he gets that

$$\log(-1) = -2 - \frac{1}{2}\cdot 4 - \frac{1}{3}\cdot 8 - \frac{1}{4}\cdot 16 - \frac{1}{5}\cdot 32 - \frac{1}{6}\cdot 64 - \cdots$$

and says that this divergent series would certainly never be zero, so $\log(-1) \neq 0$. His objection to this proof begins by looking at the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots.$$

He then considers the cases $x = -3$ and $x = 1$ which respectively give us the two sums

$$-\frac{1}{2} = 1 + 3 + 9 + 27 + 81 + 243 + \cdots$$

and

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

He points out that adding these expansions term by term shows us that

$$0 = 2 + 2 + 10 + 26 + 82 + 242 + \cdots.$$

so he sees no absurdity in the idea that 0 could be the sum of the series

$$-2 - \frac{1}{2} \cdot 4 - \frac{1}{3} \cdot 8 - \frac{1}{4} \cdot 16 - \frac{1}{5} \cdot 32 - \frac{1}{6} \cdot 64 - \cdots.$$

Euler performs these proofs and objections just to show that the great debate between his teacher and the revered Leibniz could have continued for a long time. He then points out that whether we accept that $\log(-1) = 0$ or $\log(-1) \neq 0$ we seem to arrive at insurmountable contradictions. In typical Euler fashion, just when all seems lost, he states that all of our problems can be attributed to one assumption: that each number has only one logarithm. He then goes on to prove the main theorem of the paper, that "There is always an infinity of logarithms which belong equally to each given number."

Euler proves this theorem using an ill defined infinitely small number. His lack of clarity about the nature of this infinitely small number was enough to leave many of the mathematicians of the time unconvinced of his view. Many of them, like Bernoulli, were mainly concerned with whether the logarithmic curve had two branches or not. Euler has not really answered that question outright because it was somewhat irrelevant and this left many unsatisfied with his solution.

The paper of Euler written in 1747 but not published until 1862 contained much of the same information. However, his proof of the same theorem was based on the formula $i\theta = \log(\cos\theta + i\sin\theta)$. Cajori contends that Euler was more convincing in this article and that had this article been published shortly after it had been written, the controversy might not have lasted well into the nineteenth century. [3]

After Euler's proof of the main theorem in the 1749 paper, he goes on to describe all the logarithms of arbitrary positive, negative and imaginary numbers. When he arrives at the description of the logarithms of negative numbers, the double meaning of the word ‘imaginary’ mentioned earlier comes into play. While Leibniz maintained that the logarithms of negative numbers were imaginary in the non-existent sense; Euler interpreted Leibniz’s imaginary in the $\sqrt{-1}$ sense. He says, “M. Leibniz was thus correct to maintain that the logarithms of negative numbers were imaginary.”

The is also some debate about whether Euler let his teacher John Bernoulli know about his new discoveries concerning logarithms. He had made the discovery as late as 1747, John Bernoulli died in 1748, but his first paper published on the topic was in 1751. [3]

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