On the size Ramsey number of all cycles versus a path

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Abstract

We say $G \to (\mathcal{C}, P_n)$ if $G - E(F)$ contains an n-vertex path P_n for any spanning forest $F \subset G$. The size Ramsey number $\hat{R}(\mathcal{C}, P_n)$ is the smallest integer m such that there exists a graph G with m edges for which $G \to (\mathcal{C}, P_n)$. Dudek, Khoeini and Prałat proved that for sufficiently large n, $2.0036n \le \hat{R}(\mathcal{C}, P_n) \le 31n$. In this note, we improve both the lower and upper bounds to $2.066n \le \hat{R}(\mathcal{C}, P_n) \le 5.25n + O(1)$. Our construction for the upper bound is completely different than the one considered by Dudek, Khoeini and Prałat. We also have a computer assisted proof of the upper bound $\hat{R}(\mathcal{C}, P_n) \le \frac{75}{19}n + O(1) < 3.947n$.

1 Introduction

Let $\mathcal{F}$ be a family of graphs and let H be a graph. We say that $G \to (\mathcal{F}, H)$ if every red/blue coloring of the edges of G contains a monochromatic red copy of some graph from $\mathcal{F}$ or a monochromatic blue copy of H. The *size Ramsey number* is defined as

$$\hat{R}(\mathcal{F}, H) = \min \{ |E(G)| \, : \, G \to (\mathcal{F}, H) \} .$$

In the case where $\mathcal{F} = \{F\}$, we will write $\hat{R}(F, H)$ for $\hat{R}(\mathcal{F}, H)$ and we write $\hat{R}(H)$ for $\hat{R}(H, H)$. To prove the upper bound $\hat{R}(\mathcal{F}, H) \le m$, one must prove the existence of a graph G with m edges such that $G \to (\mathcal{F}, H)$. To prove the lower bound $\hat{R}(\mathcal{F}, H) \ge m$, one must show that for every graph G on $m - 1$ edges, there is a 2 coloring which avoids both monochromatic graphs from $\mathcal{F}$ and H.

Let P_n be the path on n vertices. The size Ramsey number $\hat{R}(P_n)$ has been extensively studied, perhaps due to the fact that Erdős [9] offered \$100 for a proof or disproof of $\hat{R}(P_n) = O(n)$. Beck answered the question [3], showing that $\hat{R}(P_n) \le 900n$. After a series of improvements to the upper bound [4, 6, 12, 7] and the lower bound [3, 4, 7, 1], the state of the art is $(3.75 + o(1))n \le \hat{R}(P_n) \le 74n$ for n sufficiently large. The size Ramsey number of C_n, the cycle of length n, was first proven to be linear in n by Haxell, Kohayakawa, and Łuczak [10] with use of the sparse regularity lemma. A proof of this avoiding the use of regularity and providing explicit constants was given by Javadi, Khoeini, Omidi and Pokrovskiy [11], who proved that $\hat{R}(C_n) \le 10^6 cn$ where $c = 843$ if n is even and $c = 113482$ if n is odd. The proofs of these upper bounds as well as the best known upper bounds for $\hat{R}(P_n)$ use random (regular) graphs as their construction.

For any $c \in \mathbb{R}_+$, let $\mathcal{C}_{\le cn}$ be the family of all cycles of length at most cn and let $\mathcal{C}$ be the family of all cycles. In [5], Dudek, Khoeini and Prałat initiated the study of $\hat{R}(\mathcal{C}_{\le cn}, P_n)$ and $\hat{R}(\mathcal{C}, P_n)$.

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We remark that the parameter $\hat{R}(\mathcal{C}, P_n)$ is perhaps a natural one to study. If $G \to (\mathcal{C}, P_n)$, then G contains a path of order n after the removal of the edges of any spanning forest.

Concerning lower bounds, first note that for any $c \in \mathbb{R}_+$, $\hat{R}(\mathcal{C}_{\leq cn}, P_n) \geq \hat{R}(\mathcal{C}, P_n) \geq 2(n-1)$. The first inequality follows from the fact that any coloring of a graph which avoids all cycles in red, clearly avoids all cycles of length at most cn in red. For the second inequality, take any (connected) graph on $2(n-1)-1$ edges (and at least n vertices), color any spanning tree red, and note that there are not enough edges remaining to form a blue P_n. It is not immediately clear how one can move away from this trivial lower bound, but in [5], the authors managed to prove that for sufficiently large n and any $c \in \mathbb{R}_+$, $\hat{R}(\mathcal{C}_{\leq cn}, P_n) \geq \hat{R}(\mathcal{C}, P_n) \geq 2.00365n$.

For the upper bound, the authors of [5] use a random graph construction and techniques similar to those in [6, 7, 12] to prove that

$$\hat{R}(\mathcal{C}_{\leq cn}, P_n) \leq \begin{cases} \frac{80\log(e/c)}{c} n & \text{for } c < 1 \\ 31n & \text{for } c \geq 1 \end{cases} \tag{1}$$

Note that as $c \to 0$, this upper bound tends to infinity. It is mentioned in [5] that due to monotonicity ($m_1 \geq m_2 \implies \hat{R}(\mathcal{C}_{\leq m_1}, P_n) \leq \hat{R}(\mathcal{C}_{\leq m_2}, P_n)$), it is perhaps plausible that there is some decreasing function $\beta(c)$, such that for each fixed $c > 0$, $\hat{R}(\mathcal{C}_{\leq cn}, P_n) \sim \beta(c)n$. They mention that the "limiting case" $c \to \infty$ corresponds to $\hat{R}(\mathcal{C}, P_n)$ but they are only able to prove the upper bound $\hat{R}(\mathcal{C}, P_n) \leq \hat{R}(\mathcal{C}_{\leq n}, P_n) \leq 31n$.

In this note, we show that a significant improvement in the upper bound for $\hat{R}(\mathcal{C}, P_n)$ can be attained, not by considering the limit as c grows large, but rather by considering very *small* values of c. In fact, for our improvement, it is enough to only consider red cycles of length $3, 4$ or 5. This fact may seem surprising given the behavior of the upper bound provided in (1) as $c \to 0$, but in light of the construction we provide, the surprise diminishes. Recall that for a graph $G = (V, E)$, the kth power, G^k is a graph on vertex set V in which two vertices are adjacent if they are of distance at most k in graph G. In our main theorem, we abandon random constructions altogether and show that a very structured graph, the third power of a path, suffices.

Theorem 1.1. *Let $n \geq 2$ and let $N \geq \frac{7}{4}n + 10$. Then $P_N^3 \to (\mathcal{C}_{\leq 5}, P_n)$.*

By monotonicity, this result improves the entire range of results stated in (1).

Corollary 1.2. *For any $c \in \mathbb{R}^+$,*

$$\hat{R}(\mathcal{C}, P_n) \leq \hat{R}(\mathcal{C}_{\leq cn}, P_n) \leq \hat{R}(\mathcal{C}_{\leq 5}, P_n) \leq \frac{21}{4}n + 27.$$

Proof. The first two inequalities follow from monotonicity. Let $N = \lceil \frac{7}{4}n + 10 \rceil$. Then

$$|E(P_N^3)| = 3(N-3) + 2 + 1 = 3N - 6 \leq \frac{21}{4}n + 27.$$

□

Making use of a lemma proved with a computer check (described in Section 4), we have the following improvement.

Theorem 1.3. *Let $n \geq 2$ and let $N \geq \frac{25}{19}n + 43$. Then $P_N^3 \to (\mathcal{C}_{\leq 8}, P_n)$. Thus we have the bound*

$$\hat{R}(\mathcal{C}, P_n) \leq \hat{R}(\mathcal{C}_{\leq 8}, P_n) \leq \frac{75}{19}n + O(1) < 3.947n + O(1).$$

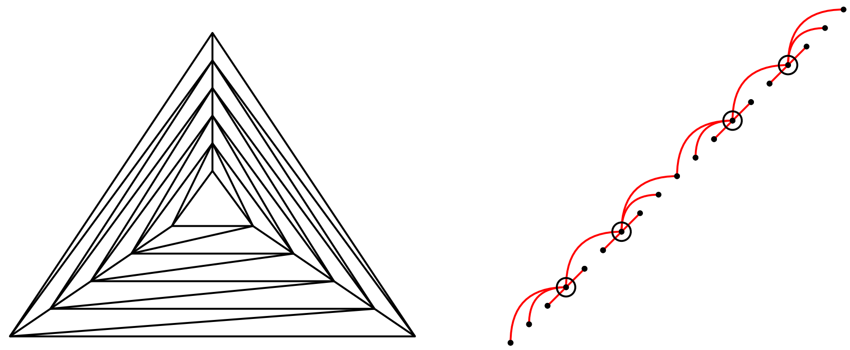

Figure 1: On the left is a planar drawing of P_N^3. On the right is a spanning tree of P_N^3 whose removal leaves behind a path of density $\sim 7/9$.

We remark that one interesting fact about P_N^3 is that it is a maximal planar graph and is in fact an *Apollonian network*. That is, it can be drawn by starting with a triangle in the plane and then repeatedly adding a new vertex inside of a current face and connecting it to each vertex of the containing face. Such a planar drawing is shown in Figure 1.

In this paper we also consider the lower bound. By improving upon the ideas in [5], we prove the following theorem.

Theorem 1.4. *Suppose n is sufficiently large and G is a graph with at most $\left(2+\frac{43}{651}\right)n - O(1)$ edges. Then there exists a red/blue coloring of $E(G)$ such that the red graph is acyclic and the blue graph contains no path of order n. Thus*

$$2.066n < \left(2+\frac{43}{651}\right)n - O(1) \leq \hat{R}(\mathcal{C}, P_n)$$

2 Proof Idea and Notation

2.1 Upper Bound

Given an integer vertex set $[N] = \{1, 2, \ldots, N\}$, we call the path with $i \sim (i+1)$ for all $i = 1, \ldots, N-1$ a *base path*. Let $N \geq \frac{7}{4}n + 10$ and let $P := P_N$ be the base path on vertex set $[N]$. Define $G := P_N^3$. We will prove that every red/blue coloring of $E(G)$ with no red C_3, C_4 or C_5 contains a blue path of order at least n.

Suppose Q is a base path on vertex set $\{0, 1, \ldots, \ell\}$ and $H = Q^3$. The *density* of a path P in H with endpoint 0 is defined as

$$r(P) := \frac{|V(P) \cap \{1, 2, \ldots, \ell\}|}{\ell}.$$

The following observation shows that one can "stitch together" paths while maintaining the density of the longer path.

Observation 2.1. *Suppose Q is a base path on vertex set $\{0, 1, \ldots, k, k+1, \ldots, k+\ell\}$ and $H = Q^3$. Suppose that P_1 is a path in $H[\{0, 1, \ldots, k\}]$ with endpoints 0 and k and $r(P_1) = d_1$, and that P_2 is a path in $H[\{k, k+1, \ldots, k+\ell\}]$ with endpoints k and $k+\ell$ and $r(P_2) = d_2$. Then $P_1 \cup P_2$ is a path in H with endpoints 0 and $k+\ell$ and $r(P_1 \cup P_2) \geq \min\{d_1, d_2\}$.*

Proof. The fact that $P_1 \cup P_2$ forms a path in H is obvious. For the density, suppose $\hat{d} = \min\{d_1, d_2\}$. Then we have

$$\begin{aligned} r(P_1 \cup P_2) &= \frac{|V(P_1 \cup P_2) \cap \{1, 2, \ldots, k+\ell\}|}{k+\ell} \\ &= \frac{|V(P_1) \cap \{1, \ldots, k\}| + |V(P_2) \cap \{k+1, \ldots, k+\ell\}|}{k+\ell} \\ &= \frac{d_1 k + d_2 \ell}{k+\ell} \geq \hat{d}. \end{aligned}$$

□

Throughout the paper, we will make use of the underlying order of the vertex set of $G = P_N^3$. Each vertex of G in $\{4, 5, \ldots, N-3\}$ has exactly 6 neighbors: $v \pm i$ where $i \in [3]$. For each vertex $v \in [N-3]$, we refer to the neighbors $v+i$, $i \in [3]$ as the *up-neighbors* of v. Given a red/blue (or R/B for short) coloring of $E(G)$, for each vertex $v \in [N-3]$, we may associate an element of $\{R, B\}^3$ (i.e. a string of length 3 with entries from $\{R, B\}$) representing the colors assigned to the edges between v and its up-neighbors. We use the notation $\text{up}(v) = c_1c_2c_3$ to mean that the edges $\{v, v+1\}, \{v, v+2\}, \{v, v+3\}$ are colored with c_1, c_2, c_3 respectively. As an illustration of this notation, we highlight one fact which we will use repeatedly without mention. If G contains no red cycles, and $\text{up}(v) = RRR$, then vertices $v+1, v+2, v+3$ form a blue triangle (else there would be a red C_3). See Figure 2.

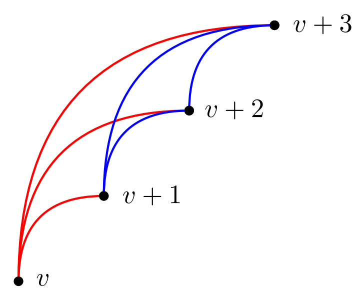

Figure 2: Blue C_3 when $\text{up}(v) = RRR$.

The main idea of the proof is to suppose that G has been R/B colored such that there is no red cycle of length at most 5 and to show that in this case, there must be a blue path of order at least n. We will find the long blue path by showing that starting at any vertex v with $\text{up}(v) \neq RRR$, one can find a blue path of density at least 4/7 in the next 10 consecutive vertices with endpoints v and w where $\text{up}(w) \neq RRR$. These short high density blue paths can then be stitched together as in Observation 2.1 to form the long blue path. The following lemma which is the main ingredient in our proof of Theorem 1.1, says that the short high density paths can always be found.

Lemma 2.2. *Let $Q = P_{11}$ on vertex set $\{0, 1, \ldots 10\}$ and let $H = Q^3$. Suppose that H has been 2-colored with no red cycles from $\mathcal{C}_{\leq 5}$. Further suppose that in H, up(0) contains at least one B. Then there is a $k \in \{1, \ldots 9\}$ such that $H[\{0, \ldots, k\}]$ contains a blue path P_B with endpoints 0 and k such that up(k) contains at least one B and*

$$r(P_B) := \frac{|V(P_B) \cap \{1, \ldots, k\}|}{k} \geq \frac{4}{7}.$$

With this lemma in hand (proved in Section 3), we can prove the main theorem.

Proof of Theorem 1.1. Let $N \geq \frac{7}{4}n + 10$, let $G = P_N^3$ and suppose that $E(G)$ has been 2-colored with red and blue such that there is no red cycle from $\mathcal{C}_{\leq 5}$. It cannot be the case that vertices 1 and 2 both have 3 red up-neighbors. Hence we may apply Lemma 2.2 starting at one of these vertices. We then repeatedly apply Lemma 2.2 to find an extension of the current blue path to another with density at least 4/7 (by Observation 2.1). We continue extending the blue path until we have found one, P_B, whose endpoint lies in $\{N-9, \dots, N\}$ (if the last blue endpoint is smaller than $N-9$, then Lemma 2.2 can be applied again). Then since $r(P_B) \geq 4/7$, we have

$$|V(P_B)| \geq \frac{4}{7} \cdot (N - 11) + 1 \geq n$$

where we have used $N - 11$ since P_B may start at vertex 2 and the additional 1 accounts for the very first vertex of P_B. □

The largest blue path density one could hope for in P_N^3 is 7/9 since we may color the edges red in a repeating pattern as indicated by Figure 1 . At most 2 of the circled vertices may be used in a blue path (as endpoints) since they would have blue degree 1. Thus we have the following.

Observation 2.3. *The best upper bound that one could ever prove using the cube of a path is* $\hat{R}(\mathcal{C}, P_n) \leq \frac{9}{7}n \cdot 3 + O(1) \approx 3.857n + O(1)$.

2.2 Lower Bound

In order to improve the lower bound, we must show that every graph G with at most $(2+\alpha)n$ edges contains a forest whose removal destroys all the paths of order n. One approach to accomplish this is to find a forest which contains many vertices of full degree (that is, vertices with the same degree in the forest as in the graph G). Such full degree vertices cannot be used in a blue path. This is the approach taken in [5]. One snag is that it is not so simple to find such forests in graphs with unbounded degree. The proof of Theorem 1.4 shows how to deal with high degree vertices and also gives an improved approach for bounded degree graphs than the one in [5].

2.3 Notation and outline

We use $N(v)$ to refer to the open neighborhood of vertex v. For two subsets X, Y of vertices, we use $e(X, Y)$ to represent the number of edges with one endpoint in X and one in Y. In Section 3, we deal with a graph on vertex set $\{0, 1, \dots 10\}$ and since we do not refer to vertex 10 in the proof, we choose to omit commas when naming paths and cycles. For example the path $(0, 1, 3, 4)$ will be denoted by 0134 and the cycle on those same vertices will be denoted (0134).

In Section 3 we prove Lemma 2.2. In Section 4 we briefly describe the computer assisted improvement to Lemma 2.2 which implies Theorem 1.3. In Section 5 we prove Theorem 1.4.

3 Proof of Main Lemma

3.1 A warm-up: density 1/3

In this subsection, to give a flavor of the proof to come, we prove a version of Lemma 2.2, replacing 4/7 with 1/3. We split into 7 cases depending on up(0). Note by assumption, we do not consider the case up(0) $= RRR$. Our goal in each case is to find a blue path P_B with density $r(P_B) \geq 1/3$ such that the non-0 endpoint, k, has a blue up-neighbor. We also note that at the end of this short warm-up we will already have proved that $P_{3n+5}^3 \to (\mathcal{C}, P_n)$ which implies $\hat{R}(\mathcal{C}, P_n) \leq 9n + O(1)$, a decent improvement over the previously known upper bound of $31n$.

- **Case 1** (up(0) $= BRR$). If up(1) $= RRR$, then edges $02, 03, 12$ and 13 are all red and so (0213) would form a red C_4, a contradiction. Thus up(1) must contain at least one B and we can take $P_B = 01$ which satisfies $r(P_B) = 1$.
- **Case 2** (up(0) $= RBR$). If up(2) contains a B, then we could take $P_B = 02$ which has $r(P_B) = 1/2$. Otherwise we can assume up(2) $= RRR$. In this case edge 12 must be blue, otherwise (0123) forms a red cycle. Edge 13 must be blue, otherwise (013) forms a red cycle. Thus we may take $P_B = 0213$ which has $r(P_B) = 1$. Note that up(3) contains a B as depicted in Figure 2 (with $v = 2$).
- **Case 3** (up(0) $= RRB$). If up(3) contains a B, then we can take $P_B = 03$. Otherwise we can assume up(3) $= RRR$. If edge 23 is red, then the red graph on vertices $\{0, 1, 2, 3, 4, 5, 6\}$ forms a tree, and so any uncolored edges must be blue. So in this case we may take $P_B = 0314$. Else we may suppose that edge 23 is blue. One of the edges 24 or 25 must be blue, otherwise (2435) is a red cycle. So then we can take $P_B = 0324$ or $P_B = 0325$.
- **Case 4** (up(0) $= BBR$). One of the edges $12, 23$ or 13 must be blue, otherwise (123) is a red cycle. So then we can take $P_B = 01$ or $P_B = 02$.
- **Case 5** (up(0) $= BRB$). One of the edges $13, 14$ or 34 must be blue, otherwise (134) is a red cycle. So then we can take $P_B = 01$ or $P_B = 03$.
- **Case 6** (up(0) $= RBB$). One of the edges $23, 24$ or 34 must be blue, otherwise (234) is a red cycle. So then we can take $P_B = 02$ or $P_B = 03$.
- **Case 7** (up(0) $= BBB$). One of the edges $12, 23$ or 13 must be blue, otherwise (123) is a red cycle. So then we can take $P_B = 01$ or $P_B = 02$.

3.2 Proof of Lemma 2.2: density $4/7$

Proof of Lemma 2.2. The proof is essentially a more intricate version of the one that appears above. Cases 3 and 6 are much more involved than the other cases so the reader may wish to read those last. We provide `python` code at the url `http://msuweb.montclair.edu/~bald/research.html` which can help with the verification of this proof.

- **Case 1** (up(0) $= BRR$)

  If up(1) $= RRR$, then edges $02, 03, 12$ and 13 are all red and so (0213) would form a red C_4, a contradiction. Thus up(1) must contain at least one B and we can take $P_B = 01$ which satisfies $r(P_B) = 1$.

- **Case 2** (up(0) $= RBR$)

  Suppose edge 12 is red. Then 023 is a blue path since edge 23 must be blue (else (0123) is a red cycle). If up(3) $= RRR$, then edge 14 must be blue (else (0143) is a red cycle), and so we can take $P_B = 02314$ since 4 has blue up-neighbors 5 and 6 and $r(P_B) = 1$. Otherwise up(3) contains a B and we can take $P_B = 023$ which satisfies $r(P_B) = 2/3$.

  Now, suppose edge 12 is blue. In this case, 0213 is a blue path (edge 13 must be blue otherwise (013) is a red cycle). If up(3) contains a B, then we may take $P_B = 0213$. Otherwise up(3) $= RRR$. In this case, edges 14, 45 and 56 are all blue. Thus we can take $P_B = 02145$ where $r(P_B) = 4/5$.

- **Case 3** (up(0) $= RRB$)

  Suppose edge 23 is red. Then edge 13 must be blue (else (0132) is a red cycle) and so 031 is a blue path.

  If edge 14 were red, then edge 24 must be blue (else (0142) is a red cycle) and so 03124 is a blue path. If up(4) contains a B, then we may take $P_B = 03124$. If up(4) $= RRR$, then 320145 is a red path, and so any other edge among these vertices must be blue. Thus we may take $P_B = 03425$ since vertex 6 is a blue up-neighbor of vertex 5 and $r(P_B) = 4/5$.

  If edge 14 were blue, then 0314 is a blue path. If up(4) contains a B, then we may take $P_B = 0314$ which has $r(P_B) = 3/4$. Otherwise, suppose up(4) $= RRR$ (which recall implies that vertices $5, 6$ and 7 form a blue triangle).

  If edge 24 were red, then edges 34 and 25 must be blue (else we have red cycles (234) or (245) respectively). Thus we may take $P_B = 034125$ since vertex 6 is a blue up-neighbor of vertex 5.

  So we assume edge 24 is blue. If edge 35 is red then edge 25 must be blue (else (235) is a red cycle). Thus we may again take $P_B = 031425$. So assume that edge 35 is blue. In this case, we have 03567 is a blue path. Now if up(7) contains a B, when we may take $P_B = 03567$ which has $r(P_B) = 4/7$ (this specific case is illustrated in Figure 3 just as an example). Otherwise if up(7) $= RRR$, then edge 68 is blue (else (4687) is a red C_4). In this case we may take $P_B = 0357689$ which has $r(P_B) = 2/3$.

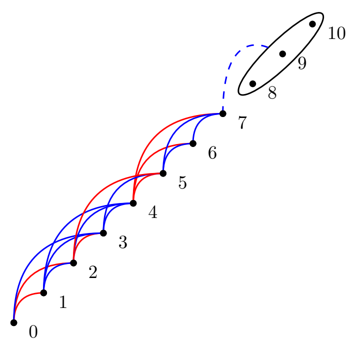

Figure 3: An illustration of the situation when the proof has led us to the assumptions up(0) $=$ RRB, edge 23 is red, 14 is blue, up(4) $= RRR$, 24 is blue, 35 is blue and up(7) contains a B. In this case, we take $P_B = 03567$ which has $r(P_B) = 4/7$.

  Now we assume edge 23 is blue. Then 0321 forms a blue path.

  Suppose edge 14 is red. Then edge 24 is blue (else (0142) is a red cycle) and so 0324 is a blue path. If up(4) contains a B, then we may take $P_B = 0324$. So suppose that up(4) $= RRR$. Then (567) is a blue triangle and edge 25 must be blue (else (01452) is a red cycle) and so we may take $P_B = 03256$ which has $r(P_B) = 2/3$.

  Now suppose edge 14 is blue. If up(4) contains a B, then we may take $P_B = 03214$. Else suppose up(4) $= RRR$ so that (567) forms a blue triangle. If edge 25 is blue, then we may take $P_B = 03256$ which has $r(P_B) = 2/3$. So suppose edge 25 is red.

If edge 35 is blue, then 03567 is blue path. If up(7) contains a B, then we may take $P_B = 03567$ with $r(P_B) = 4/7$. Otherwise suppose up(7) $= RRR$. Then we may take $P_B = 0357689$ which has $r(P_B) = 2/3$.

So suppose edge 35 is red. Then edge 36 is blue (else (3546) is a red C_4). So 03657 is a blue path. If up(7) contains a B, then we take $P_B = 03657$ with $r(P_B) = 4/7$. Otherwise suppose up(7) $= RRR$ and so edge 58 is blue (else (4578) is a red C_4). So we may take $P_B = 0367589$ which has $r(P_B) = 2/3$.

- **Case 4** (up(0) $= BBR$)

If up(1) contains a B, then we may take $P_B = 01$. Otherwise suppose up(1) $= RRR$. In this case, (234) is a blue C_3 and so we may take $P_B = 023$ which has $r(P_B) = 2/3$.

- **Case 5** (up(0) $= BRB$)

If up(1) contains a B, then we may take $P_B = 01$. Otherwise suppose up(1) $= RRR$ so that (234) is a blue C_3. Then 0324 is a blue path. If up(4) contains a B, then we may take $P_B = 0324$ which has $r(P_B) = 3/4$. Otherwise suppose up(4) $= RRR$ so that (567) is a blue C_3 and so that edge 25 is blue (else (1254) is a red C_4). Then we may take $P_B = 034256$ which has $r(P_B) = 5/6$.

- **Case 6** (up(0) $= RBB$)

Suppose edge 23 is blue. If up(3) contains a B, then we may take $P_B = 023$ which has $r(P_B) = 2/3$. Otherwise suppose up(3) $= RRR$. Then edges 24 and 25 cannot both be red (else (2435) is a red C_4). If edge 24 blue, then we may take $P_B = 03245$ which has $r(P_B) = 4/5$. If edge 25 is blue, then we may take $P_B = 0325$ which has $r(P_B) = 3/5$.

So suppose edge 23 is red. Then edges 12 and 13 cannot both be red (else (123) is a red C_3).

First suppose both edges 12 and 13 are both blue. Then 0213 is a blue path. If up(3) contains a B, then we may take $P_B = 0213$. Otherwise up(3) $= RRR$ in which case edge 24 is blue (else (234) is a red C_3) and so we may take $P_B = 031245$.

Now suppose that exactly one of 12 or 13 is blue and the other is red. Denote the blue edge as 1α and the red edge as 1β, where $\alpha, \beta \in \{2, 3\}, \alpha \neq \beta$. Notice that the edge $\alpha\beta$ is red since this is the edge 23.

Suppose edge 14 is blue. If up(4) contains a B, then we may take $P_B = 0\alpha14$. Otherwise suppose up(4) $= RRR$.

If edge $\beta4$ is red, then the red graph on vertices $\{0, \ldots, 7\}$ forms a tree, and so any other edge on these vertices must be blue. In particular, edges $25, 35$ and 36 are blue and so we may take $P_B = 02536$ which has $r(P_B) = 2/3$.

So suppose edge $\beta4$ is blue. If edge $\beta5$ is blue, then we may take $P_B = 0\alpha14\beta56$. If edge $\beta5$ is red, then again, the red graph on vertices $\{0, \ldots, 7\}$ forms a tree, and so any other edge is blue. In particular, edge $\alpha5$ is blue and so we may take $P_B = 0\beta41\alpha56$.

Suppose edge 14 is red. Then edges 24 and 34 are both blue since the red graph on vertices $\{0, \ldots, 4\}$ forms a tree.

Suppose edge 25 is red. Then the red graph on vertices $\{0, \ldots, 5\}$ forms a tree and so any other edge on these vertices must be blue. In particular edges 35 and 45 are blue. So 02435 forms a blue path. If up(5) contains a B, then we may take $P_B = 02435$. Otherwise suppose up(5) $= RRR$, in which case we may take $P_B = 0245367$.

Suppose edge 25 is blue. If up(5) contains a B, then we may take $P_B = 02435$. Otherwise suppose that up(5) $= RRR$. If edge 46 is red, then the red graph on $\{0, \dots, 8\}$ forms a tree and so we may take $P_B = 0245367$. If edge 46 is blue, then we may take $P_B = 03467$ which has $r(P_B) = 4/7$.

- **Case 7** (up(0) $= BBB$)

  If up(1) contains a B, then we may take $P_B = 01$. Otherwise suppose up(1) $= RRR$ in which case we may take $P_B = 023$.

□

4 Computer assisted improvement

With the use of a computer program (written in `python`, making use of the `networkx` package, and made available at the url[1] `http://msuweb.montclair.edu/~bald/research.html`) we have a proof of the following lemma which finds a higher density path than Lemma 2.2.

Lemma 4.1. *Let $Q = P_{43}$ on vertex set $\{0, 1, \dots 42\}$ and let $H = Q^3$. Suppose that H has been 2-colored with no red cycles from $\mathcal{C}_{\leq 8}$. Further suppose that in H, up(0) $\notin \{RRR, RRB\}$. Then there is a $k \in \{1, \dots 39\}$ such that $H[\{0, \dots, k\}]$ contains a blue path P_B with endpoints 0 and k such that up(k) $\notin \{RRR, RRB\}$ and*

$$r(P_B) := \frac{|V(P_B) \cap \{1, \dots, k\}|}{k} \geq \frac{19}{25}.$$

Using this improved density of $19/25 = .76$, Theorem 1.3 follows just as Theorem 1.1 followed from Lemma 2.2. The algorithm proceeds much as our proof of Lemma 2.2 proceeds. Suppose up(0), $\dots$, up($k-1$) have been assigned and one finds neither a red cycle nor a blue path of the desired ratio ending at $k-1$. Then we iterate through all 8 possibilities for up(k), again searching for a red cycle or a high density blue path (ending at k) and deepening the recursion when neither is found. In order to cut down on cases checked, we forced the program to avoid the most work intensive "Case 3", hence the requirement up(0), up(k) $\notin \{RRR, RRB\}$. Note that any coloring of P_N^3 with no red cycles must satisfy $\{\text{up}(0), \text{up}(1)\} \not\subseteq \{RRR, RRB\}$ and so this is an okay assumption. As a demonstration of the growth of complexity, we mention that the output of the program which verifies a density of 4/7 (i.e. equivalent to the proof of Lemma 2.2) is a `.txt` file of size 85 KB. The file which verifies the density of 3/4 has size 1.7 MB and the file which verifies the density of 19/25 has size 34 MB. As discussed in Section 2.1, the best density one could hope for in P_N^3 is $7/9 \approx 0.7777$. Due to our proof method (stitching together segments), it is unlikely that our program (as currently written) will be able to prove the exact bound of 7/9; one can color the portion near vertex 0 'badly' in a way that lowers the overall density of the segment.

5 Lower Bound

In this section we prove Theorem 1.4 by improving on the ideas which appear in [5]. The following reduction lemma essentially appears as a lemma in [1]. In that paper, the lemma concerns avoidance monochromatic paths in both colors rather than cycles in red and a path in blue. However, the proof is almost identical, so we have decided to omit it. This lemma allows us to concentrate on graphs with minimum degree at least 3.

[1] A `.txt` file containing the output of the program is also available.

Lemma 5.1. *Let n be a positive integer with $n \geq 6$. If every connected graph with at most m edges and minimum degree at least 3 has a 2-coloring such that the red graph is acyclic, and every blue path has order less than $n-2$, then every graph with at most m edges has a 2-coloring such that the red graph is acyclic and every blue path has order less than n.*

We also make use of the following lemma which shows how to find a forest in a bounded degree graph whose removal creates many vertices of degree 0 or 1 (thus unsuitable for paths in the remaining graph).

Lemma 5.2. *Suppose G is connected and has n vertices and maximum degree Δ. Then G contains a forest F and disjoint subsets $A_0, A_1 \subseteq V(G)$ such that*

(i) $A_0 \cup A_1$ *is an independent set*

(ii) $d_F(v) = d_G(v)$ *for all* $v \in A_0$

(iii) $d_F(v) \geq d_G(v) - 1$ *for all* $v \in A_1$

(iv) $|A_0| + \frac{1}{2}|A_1| \geq \gamma_\Delta n$ *where*

$$\gamma_\Delta = \left(\frac{1}{\Delta^2 + \Delta + 2} + \frac{3}{2(\Delta^2 + 2\Delta + 3)} \right).$$

Proof. We greedily build the forest F and maintain disjoint sets A_0, A_1, X, Y. Throughout, $X = N(A_0 \cup A_1)$ and $Y = V(G) \setminus (A_0 \cup A_1 \cup X)$, and so there are no edges between Y and $A_0 \cup A_1$. Initialize $A_0, A_1, X, F = \emptyset$ and $Y = V(G)$.

We start with *Phase 1*. Begin by adding an arbitrary vertex to A_0, removing it from Y and updating X. At each subsequent step of Phase 1, we look for a vertex $v \in Y$ such that $|N(v) \cap X| \leq 1$. If such a vertex v exists, we add v to A_0, add all of v's incident edges to F, and include all of v's neighbors in X. When no such vertex v exists, then Phase 1 ends. At the end of Phase 1, every vertex in Y has at least 2 neighbors in X and every vertex in X has at most $(\Delta - 1)$ neighbors in Y (since each vertex in X has a neighbor in A_0), so $2|Y| \leq e(X, Y) \leq (\Delta - 1)|X|$ and also $|X| \leq \Delta |A_0|$. So at the end of Phase 1

$$n = |A_0| + |X| + |Y| \leq |A_0| + \Delta|A_0| + \frac{\Delta - 1}{2}\Delta|A_0| = \left(\frac{\Delta^2 + \Delta + 2}{2}\right)|A_0|$$

so $|A_0| \geq \frac{2}{\Delta^2+\Delta+2}n$.

In *Phase 2* we add vertices to A_1 which have $|N(v) \cap X| \leq 2$. If there is a vertex with $|N(v) \cap X| \leq 1$, we handle it as above. If no such v exists, then we next look for a vertex $v \in Y$ such that $|N(v) \cap X| = 2$. In this case, we move v to A_1, we add to F, any edges incident to v and not X. Of the two edges incident to both v and X, we arbitrarily choose one to add to F. If no such v exists, we terminate the process. At the end of Phase 2, every vertex in Y has at least 3 neighbors in X.

By construction, one can observe that $A_0 \cup A_1$ remains independent since we only add vertices from Y. Also by construction, F remains a forest and the degree conditions in (ii) and (iii) are met. It remains to show that at the end of the process, $|A_0| + \frac{1}{2}|A_1| \geq \gamma_\Delta n$.

Note that $|X| \le \Delta|A_0 \cup A_1|$ and that at the end of Phase 2, we have $3|Y| \le e(X,Y) \le (\Delta-1)|X|$. Thus at termination, we have $|Y| \le \frac{\Delta-1}{3}|X|$ and so

$$\begin{aligned} n = |A_0 \cup A_1| + |X| + |Y| &\le |A_0 \cup A_1| + \frac{\Delta+2}{3}|X| \\ &\le |A_0 \cup A_1| + \frac{\Delta+2}{3} \cdot \Delta|A_0 \cup A_1| \\ &= \frac{\Delta^2+2\Delta+3}{3}|A_0 \cup A_1| \end{aligned}$$

and so $|A_0| + |A_1| \ge \frac{3}{\Delta^2+2\Delta+3}n$. To finish, we observe

$$\begin{aligned} |A_0| + \frac{1}{2}|A_1| &= \frac{1}{2}|A_0| + \frac{1}{2}(|A_0| + |A_1|) \\ &\ge \left(\frac{1}{2}\frac{2}{\Delta^2+\Delta+2} + \frac{1}{2}\frac{3}{\Delta^2+2\Delta+3}\right) n = \gamma_\Delta n. \end{aligned}$$

□

Proof of Theorem 1.4. Suppose $G=(V,E)$ is connected, has $e=(2+\alpha)n$ edges and $G \to (\mathcal{C}, P_n)$. In light of Lemma 5.1, we also assume that $\delta(G) \ge 3$. We note that technically, by using Lemma 5.1, we should now change our goal to finding a coloring such that the red graph is acyclic and every blue path is of order less than $n-2$. For readability, we continue to forbid paths of order n and mention that the $O(1)$ in the statement of Theorem 1.4 takes care of the issue. We may assume that G has $N=(1+\beta)n$ vertices where $\beta \le \alpha$ (else we may take any spanning tree, color it red and note that there are too few remaining edges to have a blue path of order n).

Let X be the set of vertices of degree at least 4. Then $|X| \ge 2n - N$. To see this, note that by the assumption $G \to (\mathcal{C}, P_n)$, G must have a path of order n and we may color its edges red (which is acyclic in red). Then the uncolored edges must have a path of order n (otherwise we could color them all blue). Thus we have two edge-disjoint paths P_1, P_2 on vertex sets A_1, A_2, each of size n, and any vertex in $A_1 \cap A_2$ has degree at least 4. Thus we have $|X| \ge |A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2| \ge 2n - N$.

Let B be the set of vertices of degree at least $d+1$ (we will end up taking $d=5$). Then

$$\begin{aligned} (4+2\alpha)n = 2e = \sum_v d(v) &\ge (d+1)|B| + 4(|X|-|B|) + 3(N-|X|) \\ &= (d-3)|B| + |X| + 3N \\ &\ge (d-3)|B| + 2n + 2N \end{aligned}$$

and so rearranging, we have

$$|B| \le \frac{1}{d-3}\left((2+2\alpha)n - 2N\right) = \frac{2}{d-3}\left((1+\alpha)n - N\right) = \frac{2}{d-3}(\alpha-\beta)n.$$

Let

$$\gamma_d := \left(\frac{1}{d^2+d+2} + \frac{3}{2(d^2+2d+3)}\right).$$

Note that $G[V \setminus B]$ has maximum degree d and so we may apply Lemma 5.2 to each component of G in order to find a forest F and sets A_0, A_1 with

$$|A_0| + \frac{1}{2}|A_1| \ge \gamma_d \cdot (N - |B|).$$

We color all edges in F with red, complete this forest to a red tree in G and then color the remaining edges in G with blue. Let $R = V \setminus (A_0 \cup A_1 \cup B)$. So every vertex in A_0 has only red edges to R and every vertex in A_1 has at most one blue edge to R. Suppose $P = (v_1, v_2, \ldots, v_k)$ is a blue path. Note that if $v_i \in A_0$ for some $1 < i < k$, then v_{i-1} and v_{i+1} must both be in B. Also, if $v_i \in A_0$ for some $1 < i < k$, then at least one of v_{i-1} and v_{i+1} is in B. For $X \in \{A_0, A_1, B, R\}$, let $X' = V(P) \cap X$. So if we let $e_P(A_0 \cup A_1, B)$ count the number of edges in P with one end in $A_0 \cup A_1$ and the other end in B, we have $2|A_0'| + |A_1'| - 2 \leq e_P(A_0 \cup A_1, B) \leq 2|B'|$, and so $|A_0'| + |A_1'| \leq |B'| + \frac{1}{2}|A_1'| + 1$. We then have

$$\begin{aligned} |V(P)| &= |R'| + |A_0'| + |A_1'| + |B'| \\ &\leq |R| + \frac{1}{2}|A_1'| + 2|B'| + 1 \\ &\leq N - |A_0| - |A_1| - |B| + \frac{1}{2}|A_1'| + 2|B'| + 1 \\ &\leq N - |A_0| - \frac{1}{2}|A_1| + |B| + 1. \end{aligned}$$

We see that if $N - (|A_0| + \frac{1}{2}|A_1|) + |B| < n - 1$, then there is no blue path of order n.

$$\begin{aligned} \frac{1}{n-1}\left(N - (|A_0| + \frac{1}{2}|A_1|) + |B|\right) &\leq \frac{1}{n-1}\left(N - \gamma_d\left(N - |B|\right) + |B|\right) \\ &= \frac{1}{n-1}\left((1-\gamma_d)N + (1+\gamma_d)|B|\right) \\ &\leq (1-\gamma_d)(1+\beta) + (1+\gamma_d)\cdot\frac{2}{d-3}(\alpha - \beta) + O(1/n) \\ &= (1-\gamma_d) + \beta\left(1 - \gamma_d - \frac{2}{d-3}(1+\gamma_d)\right) \\ &\qquad + \alpha\left(\frac{2}{d-3}\right)(1+\gamma_d) + O(1/n). \end{aligned}$$

We set

$$f(\alpha, \beta, d) := (1-\gamma_d) + \beta\left(1 - \gamma_d - \frac{2}{d-3}(1+\gamma_d)\right) + \alpha\left(\frac{2}{d-3}\right)(1+\gamma_d).$$

This function is decreasing in β for $d = 4, 5$ and increasing in β for $d \geq 6$. When $d = 5$, we may maximize this function by setting $\beta = 0$, and in this case we get

$$f(\alpha, 0, 5) = \frac{651}{608}\alpha + \frac{565}{608}.$$

So we have that $N - (|A_0| + \frac{1}{2}|A_1|) + |B| < n - 1$ whenever $\alpha < \frac{43}{651} - \Omega(1/n)$. One can check that using $d = 4, 6$ yields the bounds $\alpha < 5/109$ and $\alpha < 39/709$ (recalling that for $d = 6$, one must set $\beta = \alpha$ when maximizing) both of which are worse than $43/651$. For all $d \geq 7$, the bound is also worse. $\square$

6 Concluding Remarks

In this paper we have considered the size Ramsey number for the family of cycles versus a path of order n. In contrast to many recent results on size Ramsey numbers of paths and cycles, we use a non-random construction. This, however, is due to the fact that the question considered included forbidden short cycles. We note in passing that by considering the third power of a cycle C_N^3 with $N = \frac{25}{19}n + O(1)$, our proof easily implies that

$$\hat{R}(\mathcal{C}_{\leq 8}, \mathcal{C}_{\geq n}) \leq 3.947n$$

where $\mathcal{C}_{\geq n}$ is the family of all cycles of length at least n.

The most obvious open problem is to close the gap between the lower bound of $2.066n$ and the upper bound of $3.947n$. It is possible that there is a nice proof that every two coloring of P_N^3 contains a blue path of density 7/9, but we were unable to find one.

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