

COLORING GRAPHS WITH GRAPHS: A SURVEY

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1. INTRODUCTION

Many problems in and areas of graph theory can be thought of in terms of graph homomorphisms. For graphs G and H , a *homomorphism* from G to H is a function $\phi : V(G) \rightarrow V(H)$ such that if $xy \in E(G)$ then $\phi(x)\phi(y) \in E(H)$. (See [1] for a comprehensive introduction.) Let $\text{Hom}(G, H)$ be the set of homomorphisms from G to H . In this survey, we will be interested in counting homomorphisms from G to H and so we let $\text{hom}(G, H) = |\text{Hom}(G, H)|$. When G is small relative to H , then homomorphisms from G to H roughly correspond to embeddings of G inside of H . We will focus on the case when G is large and H is small. In this case, we can think of a homomorphism from G to H as a labeling of the vertices of G with the vertices of H . Thus, we sometimes refer to a homomorphism from G to H as an *H -coloring* of G . Since the labeling corresponds to a homomorphism, vertices in G that are adjacent can only receive labels of vertices that are adjacent in H . We will want to allow that adjacent vertices in G receive the same label, and so permit the image graph H to have loops. We will assume throughout this survey that the pre-image graph G is simple, and so has no loops or multiple edges.

Let us consider some examples of H -colorings of graphs for various H .

Example 1: Homomorphisms from G to K_q correspond to proper q -colorings of the vertices of G .

Example 2: If $H = K_q^\circ$, a complete graph with q vertices all of which are looped, then the set of homomorphisms from G to H consists of all maps $f : V(G) \rightarrow [q] = \{1, 2, \dots, q\}$. In other words, there are no restrictions in a K_q° -coloring of G .

Example 3: Homomorphisms from G to H_I , the graph constructed from P_2 (the path on two vertices) by adding a loop to one of the vertices, correspond to independent sets in G . This is because if we label the vertices H_I as in Figure 1 and $\phi \in \text{Hom}(G, H_I)$, then $\phi^{-1}(a)$ forms an independent set and there are no restrictions on vertices in $\phi^{-1}(b)$.



FIGURE 1. The graph H_I .

An interesting class of problems arises when the image graph H is fixed and G is allowed to vary over some set of graphs \mathcal{G} . We can then ask which graph(s) $G \in \mathcal{G}$ maximize (or minimize) $\text{hom}(G, H)$. These problems will be the main focus of this survey. In Section 2, we let the image graph be H_I , and so we are trying to maximize (or minimize) the number of independent sets over \mathcal{G} . In Section 3, we investigate other image graphs.

2. INDEPENDENT SETS

As noted above, this section will outline many results related to maximizing or minimizing the number of independent sets in a graph subject to some constraint(s). We let $\mathcal{I}(G)$ be the set of independent sets in G and $i(G) = |\mathcal{I}(G)|$, so that $i(G) = \text{hom}(G, H_I)$. We also will occasionally be interested in results involving independent sets of a fixed size and so let $\mathcal{I}_t(G) = \{I \in \mathcal{I}(G) : |I| = t\}$ and $i_t(G) = |\mathcal{I}_t(G)|$.

2.1. Various constraints. Our first result gives the maximum and minimum number of independent sets among trees of order n . (For a graph $G = (V, E)$, we let the *order* of G be $|V|$ and the *size* of G be $|E|$.) Let $\mathcal{T}(n)$ be the set of trees with n vertices. Prodinger and Tichy [2] were able to show the following.

Theorem 2.1. *If $T \in \mathcal{T}(n)$, then*

$$i(P_n) \leq i(T) \leq i(K_{1,n-1}).$$

Prodinger and Tichy referred to $i(G)$ as the *Fibonacci number of G* since $i(P_n) = F_{n+2}$, the $(n+2)^{\text{nd}}$ Fibonacci number (where $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$). Alameddine [3] proved a similar result for maximal outerplanar graphs. Recall that the *join* of graphs G and H , denoted $G \vee H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. Also, the *square of a graph G* , denoted G^2 , has vertex set $V(G)$ and edges between any pair of vertices at distance at most two in G .

Theorem 2.2. *If G is a maximal outerplanar graph on n vertices, then*

$$i(P_n^2) \leq i(G) \leq i(K_1 \vee P_{n-1}).$$

Pedersen and Vestergaard [4] considered the class of unicyclic graphs. Following their notation, we let $H_{n,k}$ be the graph formed by adding $n - k$ leaves to one vertex of a cycle of length k .

Theorem 2.3. *If G is a unicyclic graph on n vertices whose cycle is of length k , then*

$$i(C_n) \leq i(G) \leq i(H_{n,k}).$$

Hua [5] considered the problem of maximizing the number of independent sets over 2-edge-connected graphs.

Theorem 2.4. *If G is a 2-edge-connected graph, then*

$$i(G) \leq i(K_{2,n-2}).$$

Hua and Zhang [6] were able to give an upper bound for graphs with a given number of cut-vertices. We let $S_{n,k}$ be the graph formed from $K_{1,k}$ by subdividing one edge $n - k$ times.

Theorem 2.5. *If G is a graph on n vertices with $k \geq 1$ cut-vertices, then*

$$i(G) \leq i(S_{n,k}).$$

While there are many other results of this type, let note one other natural constraint before moving to bounds on the number of edges in the graphs in question. The following result bounds the independence number of the graphs under consideration, say that $\mathcal{G}_\alpha(n)$ consists of graphs on n vertices with $\alpha(G) \leq \alpha$. A lower bound in this case is easy: take an independent set of size α and add all other edges, yielding $E_\alpha \vee K_{n-\alpha}$. (See [7] for a proof.) The upper bound is less trivial and has, in fact, been rediscovered many times. We let $T_{n,\alpha}$ be the Turán graph with α parts and \overline{G} denote the complement of G .

Theorem 2.6. *If G is a graph on n vertices with $\alpha(G) \leq \alpha$, then*

$$i(G) \leq i(\overline{T_{n,\alpha}}).$$

In fact, independent sets of any fixed size are maximized by $\overline{T_{n,\alpha}}$. This result is originally due to Zykov [8] (who solved it in the context of complete subgraphs), but proved independently by many others including Erdős [9], Sauer [10], Hadžiivanov [11], and Roman [12]. A short proof can be found in [13]. In fact, these results give bounds on the number of cliques of a particular size in a graph with given clique number.

2.2. Regular graphs. The question of maximizing the number of independent sets in regular graphs has attracted much attention because of its applications to other fields. For example, independent sets in the Cayley graph of a group correspond to sum-free sets in the group and Cayley graphs are regular. Granville [14] conjectured that if G is a d -regular graph on n vertices, then

$$i(G) \leq 2^{(1/2+\varepsilon(d))n},$$

where $\varepsilon(d) \rightarrow 0$ as $d \rightarrow \infty$. Alon [15] settled this conjecture as follows.

Theorem 2.7. *If G is a d -regular graph on n vertices, then*

$$i(G) \leq 2^{(1/2+O(d^{-0.1}))n}.$$

Kahn [16] used a beautiful entropy argument to improve Alon's result for bipartite graphs.

Theorem 2.8. *If G is a d -regular bipartite graph on n vertices, then*

$$i(G) \leq (2^{d+1} - 1)^{n/(2d)}.$$

It should be noted that this result is sharp when $2d$ divides n by taking $n/(2d)$ disjoint copies of $K_{d,d}$. Kahn conjectured that this bound could be extended to all regular graphs, rather than just bipartite ones. Nearly a decade later, Zhao [17] settled this conjecture in the affirmative using the often rediscovered bipartite double cover of a graph.

The question of minimizing the number of independent sets in a regular graph is also of interest. Recently, Radcliffe and the author [18] were able to show a lower bound that corresponds to the upper bound of Kahn and Zhao.

Theorem 2.9. *If G is a d -regular graph on n vertices, then*

$$i(G) \geq (d+2)^{n/(d+1)}.$$

This bound is also sharp: when $(d+1)|n$, it is achieved by the disjoint union of $n/(d+1)$ copies of K_{d+1} .

The maximization of independent sets of fixed size have also been considered in regular graphs inspired, in part, by the following conjecture of Kahn [16].

Conjecture 1. *If G is a d -regular graph on n vertices where $2d|n$, then*

$$i_t(G) \leq i_t\left(\frac{n}{2d}K_{d,d}\right),$$

for all t with $0 \leq t \leq n$.

Some partial progress on this conjecture has been made. Carroll, Galvin, and Tetali [19] were able to give an asymptotic bound. All logarithms in this paper are base two. For $\alpha \in [0, 1]$, we let $H(\alpha)$ be the binary entropy function, i.e.,

$$H(\alpha) = -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha),$$

where we let $0 \log 0 = 0$.

Theorem 2.10. *If n , d , and t are sequences satisfying $t = (\alpha n)/2$ for a fixed $\alpha \in (0, 1)$ and G is a sequence of d -regular graphs on n vertices, then*

$$\log i_t(G) \leq \begin{cases} \frac{n}{2} \left(H(\alpha) + \frac{2}{d} \right) & \text{for any } G, \\ \frac{n}{2} \left(H(\alpha) + \frac{1}{d} \right) & \text{if } G \text{ is bipartite.} \end{cases}$$

Galvin [20] was able to improve this bound for general d -regular graphs when $n = \omega(d \log d)$, $d = \omega(1)$, and $t = (\alpha n)/2$ so that it matches the conjectured bound in the first two terms in the exponent.

2.3. Graphs of fixed size. Another natural problem arises when the class of graphs under consideration consists of graphs of fixed order and size. The extremal graph in this case is defined in terms of the *lexicographic (or lex) ordering* in which we let $A < B$ if $\min(A \triangle B) \in A$. We are interested in the lex ordering on $\binom{[n]}{2} = \{A \subseteq [n] : |A| = 2\}$ in which the sets are ordered as follows:

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \dots, \{1, n\}, \{2, 3\}, \{2, 4\}, \dots, \{2, n\}, \{3, 4\}, \dots$$

Define the *lex graph with n vertices and m edges*, denoted $L(n, m)$, to be the graph with vertex set $[n]$ and edge set comprised of the first m elements of $\binom{[n]}{2}$ according to the lex ordering. The following result is a consequence of the Kruskal-Katona theorem [21, 22]. (See [23] for another proof.)

Corollary 2.11. *If G is graph with n vertices and m edges, then*

$$i(G) \leq i(L(n, m)).$$

In fact, the Kruskal-Katona theorem implies the following result about independent sets of a fixed size as well.

Corollary 2.12. *If G is a graph with n vertices and m edges and t is such that $0 \leq t \leq n$, then*

$$i_t(G) \leq i_t(L(n, m)).$$

It seems that lower bounds in the case of graphs of fixed order and size is a difficult problem. In [13], Radcliffe and the author were able to give a lower bound on the number of independent sets in a graph with n vertices and m edges provided $0 \leq m \leq n$. We define the extremal graphs $Z(n, m)$ depending on where m is in the above range. If $0 \leq m < n/2$, then $Z(n, m)$ consists of independent edges and isolated vertices. For $n/2 \leq m \leq n$, the graph $Z(n, m)$ consists mainly of independent edges and triangles, along with at most one P_3 or P_4 . If $m = n$, then $Z(n, m)$ consists of disjoint triangles, along with at most one C_4 or C_5 .

Theorem 2.13. *Let n be a positive integer and m be an integer such that $0 \leq m \leq n$. If G is a graph with n vertices and m edges, then*

$$i(G) \geq i(Z(n, m)).$$

Of course, this result leaves much to be desired. While it would be nice to settle the question entirely, we propose the following conjecture as a step towards this goal.

Conjecture 2. *If $d|n$, $m = \frac{n}{d} \binom{d}{2}$ and G is a graph with n vertices and m edges, then*

$$i(G) \geq i\left(\frac{n}{d}K_d\right).$$

It should also be noted that there has been a lot of research devoted to the question of minimizing the number of K_t s for small t in graphs of fixed order and size. Of course, this problem corresponds to minimizing $i_t(\overline{G})$ for small t (and since the size of G is fixed, so is that of \overline{G}). A survey of this area and many others related to homomorphisms can be found in [24]. Recently, Razborov [25] improved the bound for $t = 3$ and Nikiforov [26], in turn, improved this and the best known bound for $t = 4$.

For independent sets, it is natural to ask whether these extremal results extend to hypergraphs. For an r -uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ and an integer j with $1 \leq j \leq r$, we say that a set $I \subseteq V$ is j -independent if $|I \cap E| < j$ for all $E \in \mathcal{E}$. Note that if \mathcal{H} is 2-uniform (and so is therefore a graph), then a 2-independent set is what we have called an independent set above. In general, a 1-independent set in an r -uniform hypergraph \mathcal{H} is a set of isolated vertices. Thus, if we ask which r -uniform hypergraph with n vertices and m edges has the maximum number of 1-independent sets, the answer follows from maximizing the number of isolates. One way to do this is to take the r -uniform colex hypergraph with n vertices and m edges, denoted $\mathcal{C}_{(r)}(n, m)$, which has vertex set $[n]$ and edge set consisting of the first m elements according to the colex order¹ on $\binom{[n]}{r} = \{A \subseteq [n] : |A| = r\}$. If, on the other hand, we ask which r -uniform hypergraph has the maximum number of r -independent sets, then the Kruskal-Katona theorem again implies that this is done by the lex hypergraph, defined in the obvious way.

The other cases, i.e., when $1 < j < r$, of maximizing the number of j -independent sets in an r -uniform hypergraph with n vertices and m edges remain open questions. Radcliffe and the author [27] were able to give an asymptotic bound that is sharp up to an error in the exponent. We define the extremal hypergraphs as follows. Let the r -uniform hypergraph denoted $\mathcal{S}_{(r)}^j(A, B)$ have vertex set $V = A \cup B$ and edge set

$$\left\{ e \in \binom{V}{r} : |e \cap A| < j \right\}.$$

When the identity of the sets A and B is unimportant, we write $\mathcal{S}_{(r)}^j(k, n - k)$ for $\mathcal{S}_{(r)}^j(A, B)$ where $|A| = k$ and $|B| = n - k$. Let $i^j(\mathcal{H})$ be the number of j -independent sets in a hypergraph \mathcal{H} . Note that $e\left(\mathcal{S}_{(r)}^j(k, n - k)\right)$ decreases as k increases and so for fixed integers n and q (and j and r), there is a maximal k for which $q \leq e\left(\mathcal{S}_{(r)}^j(k, n - k)\right)$.

Theorem 2.14. *Given $\varepsilon > 0$ and any r -uniform hypergraph \mathcal{H} on n vertices with $\varepsilon \binom{n}{r} \leq e(\mathcal{H}) \leq (1 - \varepsilon) \binom{n}{r}$, if we let k be maximal such that*

$$e(\mathcal{H}) \leq e\left(\mathcal{S}_{(r)}^j(k, n - k)\right),$$

¹The colex order is defined by letting $A < B$ if $\max(A \triangle B) \in B$.

then

$$\log i^j(\mathcal{H}) \leq (1 + o(1)) \log \left(i^j \left(\mathcal{S}_{(r)}^j(k, n - k) \right) \right).$$

2.4. Degree conditions. Perhaps the most general conjecture involving the maximization of $i(G)$ for graphs with given degree sequence was made by Kahn; see [28]. We let $\text{iso}(G)$ be the number of isolated vertices in a graph G .

Conjecture 3. *If G is any graph, then*

$$i(G) \leq 2^{\text{iso}(G)} \prod_{uv \in E(G)} (2^{d(u)} + 2^{d(v)} - 1)^{\frac{1}{d(u)d(v)}}.$$

This would imply, for example, Zhao's extension of Theorem 2.8. Galvin [29] considered a problem more general than that of regular graphs, but not as general as Conjecture 3. The class considered by Galvin was that of graphs on n vertices with a given minimum degree and he was able to prove the following.

Theorem 2.15. *Fix $\delta > 0$. There is an $n(\delta)$ such that for all $n \geq n(\delta)$, the unique graph with n vertices and minimum degree at least δ is $K_{\delta, n-\delta}$.*

This result was presented as partial progress towards the following conjecture proposed by Galvin.

Conjecture 4. *Let n and δ be integers with $n \geq 2\delta$. If G is a graph with n vertices and minimum degree at least δ , then*

$$i(G) \leq i(K_{\delta, n-\delta}).$$

Alexander, Mink, and the author [30] were able to prove a stronger version of this for bipartite graphs. This stronger version yields a “level sets” result for independent sets of a fixed size. Note that $i_0(G) = 1$ for any graph G and $i_1(G) = n$ for any graph G on n vertices. Independent sets of size two in a graph are exactly non-edges of the graph, and so $i_2(G) = \binom{n}{2} - e(G)$. If we wanted a level sets version of Conjecture 4, then we would like to have that for any graph G with n vertices and minimum degree δ , it is the case that

$$i_t(G) \leq i_t(K_{\delta, n-\delta}).$$

However, any δ -regular graph on n vertices will have less edges than $K_{\delta, n-\delta}$, and so if G is δ -regular on n vertices, then

$$i_2(G) = \binom{n}{2} - \frac{\delta n}{2} > \binom{n}{2} - \delta(n - \delta) = i_2(K_{\delta, n-\delta}),$$

provided $2\delta < n$. This corresponds to the cases when $K_{\delta, n-\delta}$ is not δ -regular itself. We were able to show the following theorem.

Theorem 2.16. *Let n , δ , and t be integers such that $2\delta \leq n$ and $t \geq 3$. If G is a graph with n vertices and minimum degree at least δ , then*

$$i_t(G) \leq i_t(K_{\delta, n-\delta}).$$

The bipartite case of Conjecture 4 is a corollary of this theorem. Unfortunately, since the bound is not exponential, we cannot use the bipartite double cover of Zhao to extend from bipartite graphs to general ones. Many cases of Conjecture 4 remain open, although recently Engbers and Galvin [31] were able to provide more evidence in support of the conjecture.

It would be nice to have some general result that would imply Conjecture 4. One possible route to this goal would be via graphs with bounded minimum *and maximum* degree. In [30], the following was conjectured which would, in fact, imply Conjecture 4.

Conjecture 5. *If G is a graph on n vertices with minimum degree at least $\delta \geq 1$ and maximum degree at most Δ , then*

$$i(G) \leq i(K_{\delta, \Delta})^{\lceil \frac{n}{\delta + \Delta} \rceil} = (2^\delta + 2^\Delta - 1)^{\lceil \frac{n}{\delta + \Delta} \rceil}.$$

We note that it is, in fact, enough to prove Conjecture 5 for bipartite graphs since the bipartite double cover does work on this bound.

The minimization problem among graphs with n vertices and minimum degree at least δ is trivial since K_n is in the class under consideration and so, of course, has the minimum number of independent sets in the class.

3. OTHER IMAGE GRAPHS

The question of maximizing (or minimizing) the number of H -colorings for $H \neq H_I$ has been less well-studied. We begin this section by considering again graphs G which are regular and then move on to questions about graphs of fixed size.

3.1. Regular graphs. Inspired by the result of Kahn (Theorem 2.8), Galvin and Tetali [32] made the fundamental observation that independent sets are an example of a homomorphism into a small image graph. They studied the general question of maximizing the number of homomorphisms from regular bipartite graphs to *any* image graph H . Their main result is as follows.

Theorem 3.1. *If G is a d -regular bipartite graph on n vertices and H is any graph with (perhaps) loops, then*

$$\text{hom}(G, H) \leq \text{hom}(K_{d,d}, H)^{n/(2d)}.$$

They conjectured that this result extends to general regular graphs, but unfortunately, this is not true. If H is E_2° , the fully-looped empty graph on two vertices, then

$$\text{hom}(K_{d+1}, E_2^\circ) = 2 > 2^{(d+1)/2d} = \text{hom}(K_{d,d}, E_2^\circ)^{(d+1)/(2d)}.$$

However, it does seem that either K_{d+1} or $K_{d,d}$ maximize the number of homomorphisms into any image graph. Galvin conjectured as much.

Conjecture 6. *If G is a d -regular graph and H is any graph with (perhaps) loops, then*

$$\text{hom}(G, H) \leq \max \{ \text{hom}(K_{d+1})^{n/(d+1)}, \text{hom}(K_{d,d})^{n/(2d)} \}.$$

Zhao [33] has made some progress on this conjecture by classifying some graphs for which the $K_{d,d}$ -bound holds.

3.2. Fixed size. One of the most well-studied problems included in this survey involves maximizing the number of proper vertex colorings using q colors over graphs with fixed order and size. As was noted in the Introduction, this question corresponds to maximizing $\text{hom}(G, K_q)$ over graphs G of fixed order and size. The question arose independently in different settings and was first posed by Linial [34] and Wilf [35]. Lazebnik [36] solved the case when $q = 2$ as follows.

Theorem 3.2. *If G is a graph with n vertices and m edges, then*

$$\text{hom}(G, K_2) \leq \begin{cases} 2^n & \text{if } m = 0, \\ 2^{n - \lceil 2\sqrt{m} \rceil + 1} & \text{if } 0 < m \leq \lfloor n^2/4 \rfloor, \\ 0 & \text{if } m > \lfloor n^2/4 \rfloor. \end{cases}$$

Lazebnik [37] was able to show that the Turán graph $T_{n,r}$ maximizes $\text{hom}(G, K_q)$ when $q = \Omega(n^6)$. Lazebnik, Pikhurko, and Wolder [38] proved that $T_{2k,2} = K_{k,k}$ is extremal for $q = 3$ and asymptotically extremal when $q = 4$. Loh, Pikhurko, and Sudakov [39], after providing a comprehensive survey of this field, were able to use the regularity lemma to give an asymptotic answer to the cases when $q \geq 4$.

States of the Widom-Rowlinson model in statistical physics correspond to homomorphisms from a graph to the fully-looped path on three vertices, P_3° . (See Figure 2.) Because of the

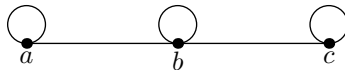


FIGURE 2. The graph P_3°

connection to the Widom-Rowlinson model, we let $\text{wr}(G) = \text{hom}(G, P_3^\circ)$. Radcliffe and the author [23] studied the problem of maximizing $\text{wr}(G)$ over graphs on n vertices and m edges. In this case, and others as we will see below, it turns out that one can prove that for any n and m , there is a threshold graph that is extremal for $\text{wr}(G)$. Threshold graphs were first introduced by Chavátal and Hammer [40]. There are many characterizations of threshold graphs (see, e.g., [41]), but the one most useful to us is the following.

Definition. A *threshold graph* is a graph that can be constructed from a single vertex graph by repeatedly adding either an isolated vertex or a dominating vertex.

Using the above characterization, we can associate a binary code of plusses and minuses with each threshold graph where plusses represent dominating vertices and minuses isolated vertices. We represent the first vertex with a dot and build the graph from the right of the code. It turns out that the lex graph is a threshold graph whose code is of the form

$$+ + \cdots + - - \cdots - (+) - - \cdots - \bullet,$$

where the parenthetical plus can occur anywhere in the string of minuses. Note that plusses in the code are adjacent to everything to their right and plusses to their left, while minuses are only adjacent to plusses to their left. The main result of Radcliffe and the author is as follows.

Theorem 3.3. *If G is a graph with n vertices and m edges, then*

$$\text{wr}(G) \leq \text{wr}(T),$$

where T is a threshold graph on n vertices and m edges with code of one of the following forms:

```

- - - - - + + + + + (-) + + + + + ●
+ + + + + + + + + - - - - - (+) - - - - - ●
+ + + + + + + + + - - - - - - - - - + + ●
+ + + + + (-) + + + - - - - - - - - - - ●
- + + + + (-) + + + - - - - - - - - - - ●
    
```

Radcliffe and the author [42] also studied the related question involving maximizing the number of homomorphisms into “the fox.” The fox, denoted F , is the graph obtained from P_3^o by deleting a loop from one of the endpoints. (See Figure 3). Let $vl(G) = \text{hom}(G, F)$.

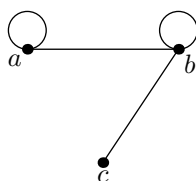


FIGURE 3. The fox, F .

We were able to obtain a similar result for $vl(G)$ as we did for $wr(G)$.

Theorem 3.4. *If G is a graph with n vertices and m edges, then*

$$vl(G) \leq vl(T),$$

where T is a threshold graph on n vertices and m edges with code of one of the following forms:

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- - - - - + + + + + (-) + + + + + ●
+ + + + + + + + + - - - - - (+) - - - - - ●
- - - - - + + + + + + + + + - - ●
+ + + + + (-) + + + - - - - - - - - - ●
- + + + + (-) + + + - - - - - - - - - ●
    
```

It is perhaps surprising that the extremal threshold graphs in Theorem 3.4 coincide with those in Theorem 3.3 except for one of the five. In that case (the third in each of the theorems), the graphs are in fact complements of one another.

Kass and the author [43] studied the case of this problem when the image graph is $E_1 \cup H_I$, which we call J . (See Figure 4.) Note that if $\phi \in \text{Hom}(G, J)$ and J is labelled as in Figure 4,

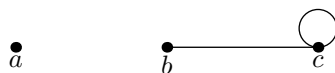


FIGURE 4. The graph J .

then $\phi^{-1}(a)$ consists of isolated vertices of G . Thus, all nontrivial components of G are mapped to the H_I component of J and, as we saw in the Introduction, the vertices mapped to b must form an independent set in G . Let $j(G) = \text{hom}(G, J)$. We were able to show, as one might expect, that the extremal graph for $j(G)$ must consist of a lex graph along with isolated vertices. The question that remains is to determine the number of isolates in the extremal graph (or, equivalently, the size of the lex component). Unfortunately, and perhaps surprisingly, if n is fixed and m is increased, the number of isolates in the extremal graph is not monotonic. However, we were able to determine the behavior when m is fixed and n is increased. For an integer m , let $\ell(m)$ be the number of vertices in the lex component of the extremal graph with $m + 1$ vertices and m edges. We proved that if $L(n', m) \cup E_{m+1-n'}$ is extremal for $j(G)$, then $L(n', m) \cup E_{n-n'}$ is extremal for any $n \geq m + 1$. Thus, the function $\ell(m)$ describes the structure of the extremal graph for $j(G)$ when the number of edges is m and the number of vertices is large enough. We were able to prove the following.

Theorem 3.5. *If m is sufficiently large, then*

$$1.414\sqrt{m} \leq \ell(m) \leq 1.897\sqrt{m} + c,$$

for a constant c .

The lower bound in the theorem corresponds to the fact that no graph on n vertices and m edges exists if $m > \binom{n}{2}$. Using computer testing, it seems that the upper bound is, in fact, closer to the truth than the lower bound.

4. CONCLUSION

Although this survey has outlined many results involving the enumeration of H -colorings of graphs, many interesting and fundamental questions remain open. Of course, the resolution of Conjecture 3 of Kahn would imply many of the results and questions in Section 2. While that may be a bit too much to hope for, we have presented other possible avenues of attack. We have also shown some extremal results for image graphs other than H_I , but there are many other image graphs left to investigate. It does not seem that minimization problems have been studied for other image graphs (including the Widom-Rowlinson model), and so this could be an area of future interest.

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