THE MULTISTEP FRIENDSHIP PARADOX

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Abstract. The Friendship Paradox, proved by Feld in 1991, states that "on average, your friends have more friends than you do." In fact, Feld proved two versions of the paradox. We discuss generalizations of each of them which talk about the average number of friends that, for instance, a friend of a friend of a friend of a friend has.

1. Introduction and Notation

The Friendship Paradox, proved by Feld [3], can be phrased as, "On average, your friends have more friends than you do." He proved two graph theoretic results, below labeled as the Friendship Paradox Theorems. We include Feld's proofs for completeness. Our notation is standard; we let

$$\overline{d} = \frac{1}{n} \sum_{v \in V(G)} d(v)$$

be the average degree of a graph G. Throughout the paper, we let $V(G)$ be the vertex set of G, $E(G)$ be its edge set, $n = |V(G)|$, and $m = |E(G)|$. We write $x \sim y$ if x is adjacent to y.

We will think of our graphs as modeling friendship networks, where we join two people if they are friends. So, the degree of a vertex is the number of friends they have. In our models, all friendships are reciprocated and so our graphs are undirected. Feld's first theorem concerns picking a (uniformly) random person, and then a (uniformly) random friend of theirs. The theorem states that this person has, on average, more friends than the initially chosen person. Friendless people are a problem for this model since the second step cannot be performed. Thus, we only consider graphs with no isolated vertices.

Friendship Paradox Theorem One. Let G be a graph with no isolated vertices. Pick a random vertex X_1 of G by first picking a uniformly random vertex X_0, then letting X_1 be a uniformly random neighbor of X_0. Then,

$$\mathbb{E}(d(X_1) - d(X_0)) = \mathbb{E}\, d(X_1) - \overline{d} \geq 0.$$

Equality holds if and only if G is locally regular, i.e., for all $u, v \in V(G)$ with $u \sim v$, we have $d(u) = d(v)$.

Proof. The equality is simply linearity of expectation. For the inequality, we bound $\mathbb{E}\, d(X_1)$ using the fact that for any positive $r \in \mathbb{R}$, we have $r + \frac{1}{r} \geq 2$, with equality if and only if $r = 1$. So,

$$\mathbb{E}\, d(X_1) = \frac{1}{n} \sum_{x_0 \in V(G)} \frac{1}{d(x_0)} \sum_{x_1 : x_1 \sim x_0} d(x_1) = \frac{1}{n} \sum_{\{x_0, x_1\} \in E(G)} \left(\frac{d(x_0)}{d(x_1)} + \frac{d(x_1)}{d(x_0)} \right)$$
$$\geq \frac{2m}{n} = \overline{d}.$$

We have equality if and only if $d(x_0) = d(x_1)$ for all $x_0 \sim x_1$, that is, G is locally regular. □

There are alternative perspectives on what it means to pick a random friend. In Feld's original paper he writes:

> The basic logic can be described simply. If there are some people with many friendship ties and others with few, those with many ties show up disproportionately in sets of friends. For example, those with 40 friends show up in each of 40 individual friendship networks and thus can make 40 people feel relatively deprived, while those with only one friend show up in only one friendship network and can make only that one person feel relatively advantaged. Thus, it is inevitable that individual friendship networks disproportionately include those with the most friends.

This describes a model in which one first picks a friendship uniformly, and then picks one of these two friends uniformly. This model exhibits the same behavior in the sense that the person eventually chosen has more friends than average. To run this model, we simply require that G has some edges. Before stating the theorem, we give an example to clarify that these two models are, in fact, different. We let $K_{a,b}$ denote the complete bipartite graph with parts of size a and b.

Example. Consider $G = K_{1,n-1}$ and let c be the central vertex of degree $n-1$. Let X_0 be a uniformly random vertex of G and X_1 be a uniformly random neighbor of X_0. Also, let Y_1 be a random vertex of G chosen by first selecting a uniformly random edge e, and then picking one of the two endpoints of e, each with probability $1/2$. Clearly, $\mathbb{P}(X_1 = c) = \frac{n-1}{n}$, whereas $\mathbb{P}(Y_1 = c) = 1/2$. See Figure 1 for a schematic.

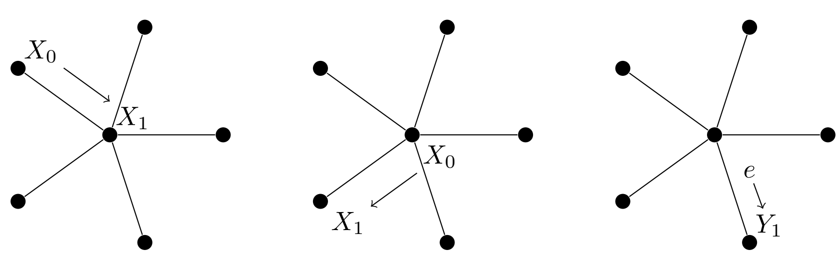

FIGURE 1. A schematic for the difference between X_1 and Y_1 on $K_{1,5}$. The first two pictures illustrate possibilities for choosing X_1; the last shows one of the possibilities for choosing Y_1.

Consider now a general graph G and again let Y_1 be a random endpoint of a random edge. If $v \in V(G)$, then $\mathbb{P}(Y_1 = v)$ is proportional to the number of edges incident with v, i.e., the degree of v. Just as in Feld's quote, vertices with higher degree are given more weight in the distribution of Y_1. Thus, it is not surprising that $\mathbb{E}\, d(Y_1) \geq \overline{d}$. On the other hand, it does not seem to us that $\mathbb{E}\, d(Y_1)$ accurately captures the notion of the average degree of a random friend, while $\mathbb{E}\, d(X_1)$ does. We are now ready to state Feld's second Friendship Paradox, that $\mathbb{E}\, d(Y_1)$ is always at least $\overline{d}$.

Friendship Paradox Theorem Two. Let G be a nonempty graph and Y_1 a random vertex of G chosen by first choosing a uniformly random edge e, and then letting Y_1 be a uniformly random endpoint of e. Then

$$\mathbb{E}\, d(Y_1) \geq \overline{d},$$

with equality if and only if G is regular.

Proof. In this proof, we need to consider both the average degree and the variance of the degrees. Recalling the definition of X_0 from the previous theorem as a uniformly randomly chosen vertex of G, we have

$$\operatorname{Var}(d(X_0)) = \frac{1}{n} \sum_{v \in V(G)} (d(v) - \overline{d})^2 = \left(\frac{1}{n} \sum_{v \in V(G)} d(v)^2 \right) - \overline{d}^2.$$

Abbreviating $\operatorname{Var}(d(X_0))$ as σ^2, we have the following:

$$\begin{aligned} \mathbb{E}\, d(Y_1) &= \frac{1}{m} \sum_{\{y_0, y_1\} \in E(G)} \frac{d(y_0) + d(y_1)}{2} = \frac{n}{2m} \cdot \frac{1}{n} \sum_{y_1 \in V(G)} d(y_1)^2 \\ &= \frac{n}{2m} \left(\overline{d}^2 + \sigma^2 \right) = \frac{\overline{d}^2 + \sigma^2}{\overline{d}} = \overline{d} + \frac{\sigma^2}{\overline{d}} \geq \overline{d}. \end{aligned}$$

The second equality follows from the fact that each vertex's degree is counted once for each edge with which the vertex is incident. Equality holds if and only if $\sigma^2 = 0$, i.e., if and only if G is regular. □

In this paper, we consider k-step generalizations of both of these theorems. Generalizing Friendship Paradox Theorem One, we consider picking a starting vertex X_0 uniformly at random and then taking a random walk $(X_0, X_1, \ldots, X_k)$. We prove that $\mathbb{E}\, d(X_k) \geq \overline{d}$. Extending Friendship Paradox Theorem Two, we consider a random sequence $(Y_0, Y_1, \ldots, Y_k)$ chosen uniformly from among all walks of length k in G. In this case we use a result of Erdős and Simonovits [2] to show that, provided k is odd, $\mathbb{E}\, d(Y_k) \geq \overline{d}$. Somewhat surprisingly, this is false in general when k is even.

1.1. The two multistep models.

We generalize both versions of Feld's theorem to k-step versions, in which we consider the number of friends (on average) that a typical friend of a friend of a … friend has. We distinguish the two models as the *random walk model* and the *homomorphism model*. Both models assign probabilities to walks in G. It will be helpful in explaining our terminology if we also define what a homomorphism between graphs is.

Definition. Given graphs G and H we say that a map $\phi : V(G) \to V(H)$ is a *homomorphism from G to H* if for all vertices $u, v \in V(G)$ we have $u \sim_G v \implies \phi(u) \sim_H \phi(v)$. In other words ϕ is an adjacency preserving map between $V(G)$ and $V(H)$.

We denote the path of length k by P_k. It will be convenient for us to let $V(P_k) = \{0, 1, 2 \ldots, k\}$, with edges between consecutive integers. A *walk in G of length k*, which we also call a *k-walk*, is a sequence of vertices $w_0w_1 \ldots w_k$ such that $w_i \sim w_{i+1}$ for $0 \leq i < k$. Equivalently, it is a homomorphism w from P_k to G that sends i to w_i. We write $\mathcal{W}_k$ for the set of all walks in G of length k.

Our first model for picking a k-walk in G is the following "random walk" model.

Definition. Let G be a graph with minimum vertex degree at least 1. We define a sequence of random variables $(X_0, X_1, X_2, \ldots)$ by picking X_0 uniformly from $V(G)$ and then at each step letting X_{i+1} be chosen uniformly from among the neighbors of X_i. This sequence is a Markov chain; in fact it is a "random walk in G" in the technical sense.

Note that this definition agrees with our notation in Friendship Paradox Theorem One. Also $\overline{d} = \mathbb{E}\, d(X_0)$. This model feels like the one that closest approximates the non-mathematical statement of the problem.

Our second model, the homomorphism model, simply picks a homomorphism $Y \in \mathcal{W}_k$ uniformly at random.

Definition. Given a graph G with at least one edge and an integer $k \geq 1$, we define a random sequence $Y = (Y_0, Y_1, Y_2, \ldots, Y_k)$ by choosing $Y \in \mathcal{W}_k$ uniformly at random.

This definition agrees with our notation in Friendship Paradox Theorem Two. When $k = 1$, the homomorphisms from P_1 to G are precisely the ordered pairs (y_0, y_1) of vertices of G such that $y_0 \sim y_1$. In other words, if Y is a uniformly random element of $\mathcal{W}_1$, then Y_1 is a (uniformly) random end of a (uniformly) random edge.

It is a standard result of Markov chain theory (see, e.g., [4]) that for connected nonbipartite graphs, $d(X_k)$ converges to $d(Y_1)$ in distribution. So in particular $\mathbb{E}\, d(X_k) \to \mathbb{E}\, d(Y_1)$ as $k \to \infty$.

2. The Multistep Friendship Paradox Theorems

In this section we prove our k-step versions of Friendship Paradox Theorems One and Two.

2.1. **The random walk model.**

To prove the result for the random walk model, we start with a simple lemma.

Lemma 1. *Given a walk $w = (w_0, w_1, \ldots, w_k) \in \mathcal{W}_k$, we define*

$$r(w) = \frac{1}{d(w_1)d(w_2)\cdots d(w_{k-1})},$$

to be the product of the reciprocal degrees of internal vertices in the walk. Then, for graphs G and all $k \geq 1$,

$$\sum_{w \in \mathcal{W}_k} r(w) = 2m.$$

Proof. Consider the sequence $X = (X_0, X_1, X_2, \ldots, X_k)$ from the random walk model. For a k-walk $w = (w_0, w_1, \ldots, w_k)$ we have

$$\mathbb{P}(X = w) = \frac{1}{n} \frac{1}{d(w_0)d(w_1)\cdots d(w_{k-1})} = \frac{1}{nd(w_0)} r(w).$$

Thus

$$\sum_{w\in\mathcal{W}_k} r(w) = \sum_{w\in\mathcal{W}_k} nd(w_0)\mathbb{P}(X=w) = n\,\mathbb{E}\,d(X_0) = n\overline{d} = 2m.$$

□

We are now ready to prove the k-step generalization of Friendship Paradox Theorem One. Note that there are more cases of equality when k is even than when $k=1$.

Theorem 2. *If G is a graph without isolated vertices and k is any nonnegative integer, then*

$$\mathbb{E}\,d(X_k) \geq \overline{d}.$$

Equality holds for k odd exactly when G is locally regular. When k is even, the requirement is that each component of G is either regular or a biregular[1] *bipartite graph.*

Proof. For a walk $w=(w_0,w_1,\dots,w_k)$, we let $\operatorname{rev}(w)=(w_k,w_{k-1},\dots,w_0)$ be the reverse of w. Since the set of reverses of k-walks is the same as the set of k-walks, we have

$$\begin{aligned}
2\,\mathbb{E}\,d(X_k) &= \sum_{w\in W_k} \left(d(w_k)\mathbb{P}(X=w) + d(w_0)\mathbb{P}(X=\operatorname{rev}(w))\right)\\
&= \sum_{w\in W_k} d(w_k)\frac{1}{n}\prod_{i=0}^{k-1}\frac{1}{d(w_i)} + d(w_0)\frac{1}{n}\prod_{i=1}^{k}\frac{1}{d(w_i)}\\
&= \frac{1}{n}\sum_{w\in W_k}\left[\frac{d(w_k)}{d(w_0)}+\frac{d(w_0)}{d(w_k)}\right]\prod_{i=1}^{k-1}\frac{1}{d(w_i)}\\
&\geq \frac{1}{n}\sum_{w\in W_k} 2\prod_{i=1}^{k-1}\frac{1}{d(w_i)}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\,d(X_k) \geq \frac{1}{n}\sum_{w\in W_k} r(w) = \frac{2m}{n} = \overline{d}.$$

Note that equality holds if and only if the endpoints of any walk of length k have the same degree. The next lemma determines exactly when this happens. □

We say that a graph is *k-walk regular* if for each walk of length k its endpoints have the same degree. For example, $K_{2,3}$ is 2-walk regular, but not 1-walk regular.

Lemma 3. *G is k-walk regular if and only if G is k'-walk regular where $k'\in\{1,2\}$ and $k'\equiv k \pmod 2$. Further, being 1-walk regular is the same as being locally regular, and G is 2-walk regular if and only if each component of G is either regular or a biregular bipartite graph.*

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[1] A bipartite graph with bipartition (A,B) is *biregular* if all vertices in A have the same degree and likewise for B.

Proof. Suppose first G is k-walk regular and that v and w are the endpoints of a k'-walk W. Then there is a k-walk with endpoints v and w obtained by traversing the first edge of W many times, so v and w have the same degree. The other direction is immediate by concatenating k'-walks.

It is clear that 1-walk regularity is equivalent to local regularity, and that if G is regular or biregular bipartite that it is 2-walk regular. Suppose then that G is 2-walk regular and, without loss of generality, connected. If G contains an odd cycle, then any two vertices of G can be joined by a walk of even length (if necessary, by traversing the odd cycle), thus G is regular. On the other hand, if G is bipartite, then any two vertices on the same side can be joined by an even walk, so must have the same degree. □

2.2. **The path homomorphism model.** It is natural to wonder whether Feld's Friendship Paradox Theorem Two behavior generalizes to $k > 1$. Here we prove that it does for odd k, and not for even k. We start with a simple lemma relating $\mathbb{E}\, d(Y_k)$ to the number of walks in G.

Lemma 4. *If G is any graph, then*

$$\mathbb{E}\, d(Y_k) = \frac{|\mathcal{W}_{k+1}|}{|\mathcal{W}_k|}.$$

Proof. For a walk $w = (w_0, w_1, \ldots, w_k) \in \mathcal{W}_k$, we let

$$\operatorname{init}(w) = (w_0, w_1, \ldots, w_{k-1}).$$

We have

$$\begin{aligned}\mathbb{E}\, d(Y_k) &= \frac{1}{|\mathcal{W}_k|} \sum_{w \in \mathcal{W}_k} d(w_k) \\ &= \frac{1}{|\mathcal{W}_k|} \sum_{w \in \mathcal{W}_k} |\{w' \in \mathcal{W}_{k+1} : \operatorname{init}(w') = w\}| \\ &= \frac{|\mathcal{W}_{k+1}|}{|\mathcal{W}_k|}.\end{aligned}$$ □

In [2], Erdős and Simonovits discuss the behavior of the parameter $v_k = |\mathcal{W}_k|/n$, the average number of walks of length k starting at a uniformly chosen vertex. Note that $v_1 = 2m/n = \overline{d}$. Their central result in this area is as follows.

Theorem 5 (Erdős-Simonovits)**.** *For all graphs G on n vertices and $k \geq 1$, we have*

$$v_k^{1/k} \geq \overline{d} = v_1^{1/1}.$$

In addition, for all $k, t \geq 1$ with k even and $k \geq t$,

$$v_k^{1/k} \geq v_t^{1/t}.$$

As was noted by Erdős and Simonovits, the first part of this theorem follows from a much more general result of Blakley and Roy [1] concerning matrices with nonnegative entries, such as the adjacency matrix of a graph. Blakley and Roy established the cases of equality which imply, in our case, that $v_k^{1/k} = \overline{d}$ for some $k > 1$ if and only if G is regular. In their proof of the second half of the theorem, Erdős and Simonovits use the convexity of $x^{k/t}$; this explains the restriction to k even.

From these results and our lemma above, we have the following.

Corollary 6. *Let G be a nonempty graph, $k \geq 1$ be odd, and $t \leq k$.*

(1) $\mathbb{E}\, d(Y_k) \geq \overline{d}$*, and*
(2) $(\mathbb{E}\, d(Y_t) \cdot \mathbb{E}\, d(Y_{t+1}) \cdots \mathbb{E}\, d(Y_k))^{1/(k-t+1)} \geq \overline{d}$*. In other words, the geometric mean of a sequence of* $\mathbb{E}\, d(Y_i)$*s, ending in an odd* i*, is at least the average degree.*

In both cases, equality holds for $k > 1$ *if and only if* G *is regular.*

Proof. Let $k \geq 1$ be an odd integer. By Theorem 5, we have

$$v_{k+1}^{1/(k+1)} = (|\mathcal{W}_{k+1}|/n)^{1/(k+1)} \geq (|\mathcal{W}_k|/n)^{1/k} = v_k^{1/k}.$$

Raising both sides to the $(k+1)^{\text{st}}$ power, we see

$$\frac{|\mathcal{W}_{k+1}|}{n} \geq \frac{|\mathcal{W}_k|}{n} \left(\frac{|\mathcal{W}_k|}{n}\right)^{1/k}.$$

Hence, using Theorem 5 again,

$$\mathbb{E}\, d(Y_k) = \frac{|\mathcal{W}_{k+1}|}{|\mathcal{W}_k|} \geq \left(\frac{|\mathcal{W}_k|}{n}\right)^{1/k} = v_k^{1/k} \geq \overline{d}. \tag{1}$$

To prove the second part, note that

$$\mathbb{E}\, d(Y_t) \cdot \mathbb{E}\, d(Y_{t+1}) \cdots \mathbb{E}\, d(Y_k) = \frac{|\mathcal{W}_{k+1}|}{|\mathcal{W}_t|},$$

and the proof proceeds similarly.

For equality, it is necessary to have equality in (1). As in the discussion after Theorem 5, this requires G to be regular. □

We end this section with an example showing that the restriction in Corollary 6 that k is odd is necessary.

Example. Consider the graph $K_{a,b} \cup K_c$. Note that

$$|\mathcal{W}_k| = \begin{cases} (a+b)(ab)^{k/2} + c(c-1)^k & \text{if } k \text{ is even,} \\ 2(ab)^{(k+1)/2} + c(c-1)^k & \text{if } k \text{ is odd.} \end{cases}$$

Note that $\overline{d} = (2ab + c(c-1))/(a+b+c)$ and so, when k is even, we have that $|\mathcal{W}_{k+1}|/|\mathcal{W}_k| \geq \overline{d}$ if and only if

$$[2(ab)^{(k+2)/2} + c(c-1)^{k+1}](a+b+c) \\ - [(a+b)(ab)^{k/2} + c(c-1)^k](2ab + c(c-1)) \geq 0.$$

This is false when $k = 2$, $a = 1$, $b = 5$, and $c = 3$. For arbitrary (even) k, if we substitute $a = t^2$, $b = t^6$, and $c = t^3$, the left hand side is a polynomial in t with leading term $-t^{4k+12}$. Thus, for large integer t, the graph $K_{t^2,t^6} \cup K_{t^3}$ has $\mathbb{E}(d(Y_k)) < \overline{d}$. This example works because the bipartite component is sensitive to parity, while the complete component is not, and the two behaviors can't mix.

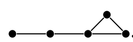
3. Further directions

In this section, we discuss possible extensions of the results in this paper. The Erdős-Simonovits Theorem (Theorem 5) has two parts: that $v_k^{1/k} \geq \overline{d}$, and a monotonicity result for the sequence $\left(v_k^{1/k}\right)_{k\geq 1}$. In fact, they made the following conjecture, extending the monotonicity part of Theorem 5.

Conjecture (Erdős-Simonovits)**.** For $k \geq t \geq 1$ both odd, we have

$$v_k^{1/k} \geq v_t^{1/t}.$$

This conjecture is still open. One might hope for corresponding monotonicity results for the sequences $(\mathbb{E}\, d(X_k))_{k\geq 0}$, which we call the *walk average degrees*, and $(\mathbb{E}\, d(Y_k))_{k\geq 0}$, the *hom average degrees* for k of fixed parity. The following examples show that we cannot expect to have monotonicity.

Example. Consider the graph . The walk average degrees are non-monotonic both for k even and for k odd. The hom average degrees are non-monotonic for k odd.

The graph $K_{1,3} \cup P_4$ has non-monotonic hom average degree for k even. We do not know an example of a connected graph for which the hom average degrees are not increasing for k even.

One other possibility for further investigation is to determine whether a version of the Friendship Paradox holds in directed graphs, perhaps better reflecting the real world situation.

4. Acknowledgments

The authors are grateful for the very helpful comments from the anonymous referees that greatly improved the paper.

The second author was sponsored by the National Security Agency under Grant H98230-15-1-0016. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

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