

CYCLE DOUBLE COVERS OF GRAPHS WITH DISCONNECTED FRAMES

JONATHAN CUTLER AND ROLAND HÄGGKVIST

ABSTRACT. Continuing work initiated by Häggkvist and Markström, we show in this paper that certain disconnected frames guarantee the existence of a cycle double cover. Specifically, we show that the disjoint union of a Kotzig and a sturdy graph forms a good frame.

1. INTRODUCTION

The cycle double cover conjecture is one of the most well known conjectures in all of graph theory. A *cycle double cover* of a graph G is a collection of (not necessarily disjoint) cycles such that each edge of G is in exactly two cycles of the collection. The cycle double cover conjecture, generally said to be due to Seymour [4] and Szekeres [5], is as follows:

Conjecture 1. *Every 2-edge-connected graph has a cycle double cover.*

While considerable effort has been put into the solution of this conjecture, it has thus far not been solved. Many different techniques have been used to attack the problem, some of which have been outlined by Jaeger [3]. Also, Zhang [6] is an excellent reference for an introduction to this field. Furthermore, Jaeger [3] proves a result stating that the cycle double cover for cubic graphs implies that for general graphs. This, of course, has meant that nearly all efforts towards the solution of the Cycle Double Cover conjecture have been focused on cubic graphs.

In this note, we continue along the same vein investigated by Häggkvist and Markström [1, 2] who introduced the idea of frames in cubic graphs to demonstrate certain classes of graphs which have cycle double covers. The definition of a frame is as follows.

Definition 2. A bridgeless cubic graph H is said to be a *frame* of a graph G if G has a spanning subgraph \hat{H} such that

- (i) \hat{H} is isomorphic to a subdivision of H , and
- (ii) the number of vertices in each component \hat{H}_i of \hat{H} has the same parity as the number of vertices in the corresponding component H_i of H .

The research initiated by Häggkvist and Markström has centered on finding graphs H so that if H is a frame of any 2-connected cubic graph G , then G has a cycle double cover. In this case, H is called a *good frame*. Further, we say that H is a *k-good frame* if any graph with H as a frame has a k -cycle double cover, where a k -cycle double cover is a cycle double cover in which the cycles can be colored with k colors in such a way that no two cycles of the same color share an edge.

Two classes of graphs which Häggkvist and Markström were able to prove were good frames are Kotzig and sturdy graphs. A proper edge coloring of a regular multigraph G is said to be *Kotzig* if the edges in each pair of color classes form a Hamiltonian cycle. A graph G is said to be *Kotzigian* or a *Kotzig graph* if it has a Kotzig coloring. We shall only be interested in 3-edge-colorable cubic Kotzig graphs, and therefore the pairs of colors will form three Hamiltonian cycles. Finally, we define a sturdy graph.

Definition 3. Let G be a 3-edge-colorable cubic graph and assume there is a 3-edge-coloring with colors 1, 2 and 3 of G such that

- (i) Edges of colors 1 and 2 form a Hamiltonian cycle.
- (ii) Edges of colors 1 and 3 form a Hamiltonian cycle.
- (iii) Edges of colors 2 and 3 form a 2-factor of G .

Further, assume that if colors 2 and 3 are exchanged on any subset of cycles of the 2-factor, we get a new proper 3-edge-coloring of G satisfying the three properties. In this case, G is said to be *sturdy*.

Armed with these definitions, let us review some the results proved by Häggkvist and Markström in [1]. While this paper will only be concerned with whether a graph is a good frame or not, many results in [1], including the following, found the k for which classes of graphs are a k -good frame.

Theorem 4. *A Kotzigian graph is a 6-good frame.*

In [1], Häggkvist and Markström also proved that graphs with a so-called *switchable* cycle double cover are 6-good frames. A graph with a switchable cycle double cover is a sturdy graph in which the 2-factor formed by colors 2 and 3 is made up of only two cycles. However, their technique is easily extendable to give the result for sturdy graphs in general.

The aim of this paper is to extend the results in the vein described above to graphs which have disconnected frames. The main result of the paper is as follows.

Theorem 5. *A disjoint union of a Kotzig graph and a sturdy graph is a good frame.*

We note that this theorem implies the following theorem which has been proved separately by the senior author.

Theorem 6. *A union of two disjoint Kotzig graphs is a good frame.*

In fact, the senior author has proven that two disjoint Kotzig graphs along with a disjoint union of arbitrarily many C_4 s also yields a good frame. Also, the same author has a proof for the disjoint union of arbitrarily many Kotzig graphs which has yet to be published.

2. PRELIMINARIES

Firstly, we present a lemma which shall be used in the proof of our main theorem. It is a rather trivial observation, but we include its proof for completeness.

Lemma 7. *Given any tree T and a subset $S \subset V(T)$ with $|S|$ even, there exists a set of paths, \mathcal{P} in T such that*

- (i) *each path has endpoints in S ,*
- (ii) *each vertex of S is an endpoint to exactly one path in \mathcal{P} , and*
- (iii) *the paths in \mathcal{P} are edge-disjoint.*

Proof. Consider all partitions \mathcal{P} of S into pairs. Note that since $|S|$ is even, \mathcal{P} is not empty. Then any $P \in \mathcal{P}$ gives a set of paths between vertices of S by simply taking the unique path in T between vertices in every pair in P . For a given $P \in \mathcal{P}$, let $\ell(P)$ be the total length of all these paths. Further, let $P' \in \mathcal{P}$ be such that $\ell(P')$ is minimal. Suppose there are then edges shared in the paths induced by this pairing of vertices of S . Then, keep each edge appearing the paths an odd number of times, and delete those which appear in an even number of paths. We are done if each edge appears in exactly one path. If not, there must be an edge in at least three paths. Take the symmetric difference of any two of these paths and we reduce the number of paths that this edge is in by two. Continue until each edge is in one path and the proof is complete. \square

In fact, we are interested in an easy corollary of Lemma 7.

Corollary 8. *In any tree T , given a set $S \subset V(T)$ with $|S|$ even, there is a subforest of T with the vertices of odd degree in T exactly the vertices in the set S .*

Proof. Take the union of the paths given by Lemma 7. □

Before we begin the proof of Theorem 5, we define various notation that will ease our explanation. For the rest of this note, we shall denote the general graph by G , which is formed by subdividing edges in our two graphs which make up the frame, and then adding a matching between the subdividing vertices. Let us call this matching M . The matching naturally partitions into three bits: edges between subdividing vertices which are both in the Kotzig graph, those between subdividing vertices in the sturdy graph and, lastly, those between the Kotzig and the sturdy graph. As will be justified below, our interest does not lie in the former two types of subdividing edges, but rather in the latter. Note that by the parity condition in the definition of a frame, we know that we have an even number of subdividing vertices in each component of our frame. We denote the Kotzig component of our frame by H_k and the sturdy component by H_s . We then color the edges of each component with colors 1, 2 and 3, with the only restriction that the colors 2 and 3 in H_s make up the non-Hamiltonian 2-factor. Let $e = xy \in M$ be an edge which goes between H_k and H_s , so that x is a subdividing vertex in H_k and y in H_s . We say that xy is an ij -edge, for $i, j \in \{1, 2, 3\}$, if the color of the edge that x subdivides in H_k is i and the color of the edge that y subdivides in H_s is j . In this case, we sometimes say that x has color i in H_k and j in H_s .

Note that we have a bit of freedom when it comes to determining which cycle of the frame a particular subdividing vertex should be on. Of course, the edge it subdivides is fixed, but each edge in H_k lies on two Hamiltonian cycles. Likewise, each edge in H_s lies on either two Hamiltonian cycles, or a Hamiltonian cycle and a cycle of the 2-factor made up of edges colored 2 and 3. Thus, we can shift around edges of M to get a particular arrangement. We do this because, as was shown in [1], if there is some arrangement so that each cycle of the 3-edge-colorings of H_s and H_k has an even number of subdividing vertices on it, then it has a cycle double cover. We state their proposition more precisely as follows.

Proposition 9. *If F is a graph with cycle double cover \mathcal{C} and G is a graph formed from F by subdividing edges an even number of times and then adding a matching M between subdividing vertices, then G has a cycle double cover if there exists an arrangement of the subdividing vertices so that each cycle C in \mathcal{C} has an even number of subdividing vertices on it.*

This is shown by taking such an arrangement and recoloring edges. Firstly, color all edges of M red. Then, color edges in the subdivided H_s and H_k green and blue by starting a subdividing vertex, coloring the edges around a cycle of the original 3-coloring say blue, then switching to green at the next subdividing vertex, and so on. This coloring is well-defined since there are an even number of switched between green and blue (since there are an even number of subdividing vertices on each cycle). Then cycles are formed by taking red/green (and red/blue) alternating walks. It is not too hard to see that these cycles form a cycle double cover of our graph G .

Before we are able to begin proving lemmas, we need to shift some edges of M around a bit so that our proofs are easier to describe. The main aim of this section is to describe our *starting configuration* of edges of M , or our default arrangement of edges of M . Unfortunately, to do this, there is quite a bit of notation that needs to be introduced. Let C_{12} be the Hamiltonian cycle in H_k formed by edges of colors 1 and 2 and likewise for C_{13} and C_{23} . Let D_{12} be the Hamiltonian cycle in H_s formed by edges of colors 1 and 2 and likewise for D_{13} . Let D_{23}^i , for $i = 1, \dots, m$ be the cycles formed by edges of color 2 and 3 in H_s and giving a 2-factor of H_s . Note that we can arrange the edges of M between H_k and H_s in a “parallel” way, i.e., we can ensure that such edges appear only between C_{12} and D_{12} , C_{13} and D_{13} and lastly C_{23} and the D_{23}^i s. We can do this since

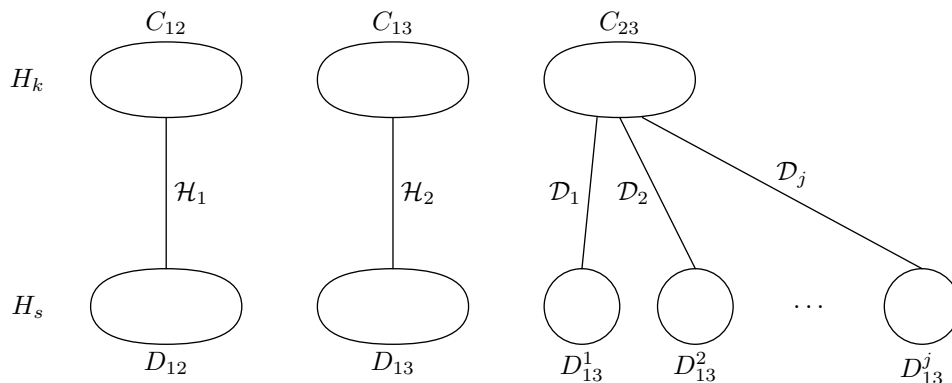


FIGURE 1. The basic setup

if we have, for example, a 13-edge in M between H_k and H_s , we can put it between C_{13} and D_{13} . After doing this, we are able to name sets of different types of edges in M . Let the edges of M between C_{12} and D_{12} be denoted \mathcal{H}_1 and likewise the edges of M between C_{13} and D_{13} be denoted \mathcal{H}_2 . Also, let the edges of M between C_{23} and D_{23}^i be denoted \mathcal{D}_i for $i = 1, \dots, j$. See Figure 1 for a picture. So each edge between H_k and H_s is in either \mathcal{H}_1 , \mathcal{H}_2 or one of the \mathcal{D}_i s. In our starting configuration, we will always minimize the edges in \mathcal{H}_1 and \mathcal{H}_2 , so that any edge of the same color in both sides which can be shifted to one of the \mathcal{D}_i s, i.e., 22- and 33-edges, will be at the start.

Proposition 9 is essentially the key to our entire proof. If we can find an arrangement of the edges of M so that C_{12} , C_{13} , C_{23} , D_{12} , D_{13} and all the D_{23}^i s have an even number of subdividing vertices on them, we are done. We will consider the edges of M depending upon where they lie. Note first that any edge of M lying entirely in H_k causes no problem whatsoever. Since H_k is made up of three Hamiltonian cycles, we know that for any edge e of M in H_k , we can think of e as being a chord in one of these Hamiltonian cycles. Thus, we can arrange edges of M in H_k so that they each contribute an even number of subdividing vertices to C_{12} , C_{13} and C_{23} . As for H_s , the same situation is true for edges of M that can be arranged to be chords of D_{12} and D_{13} .

However, the situation for the \mathcal{D}_i s is slightly more complicated. We again note that if an edge of M is a chord of any D_{23}^i , then it causes no trouble for us. So we are interested in the edges of M between the D_{23}^i s. As would be expected, we are really interested in the parity of the number of edges between any two D_{23}^i s. In order to make this clear, contract each of the D_{23}^i s to a vertex and consider the multigraph formed by these vertices and the edges between them (thus, we delete any loops formed by edges in the original cycles). Remove a maximal Eulerian subgraph of this multigraph and we are left with a forest, which we will call the D_{23} -forest from this point forward. The form of the D_{23} -forest plays a critical role in what edge sets need to be changed amongst the \mathcal{D}_i s. As will be seen later, after applying Corollary 8, we only need to ensure that each component, i.e., tree, of the D_{23} -forest has an even number of edges between it and C_{23} . In this vein, let the trees of the D_{23} -forest be denoted by T_1, T_2, \dots, T_ℓ . We shall slightly abuse this notation, using it both for the contracted trees and the cycles D_{23}^i corresponding to the vertices of this tree and edges of M between them. Then let \mathcal{T}_j for $j = 1, \dots, \ell$ be the set of edges in M between C_{23} and T_j . Now, let us describe the situation we need in order to have a so-called good configuration. Firstly, we begin by consider the D_{23}^i s in a fixed T_j . The proof will depend on switching colors in either H_k or H_s , but we need to be careful in this switching. In terms of the D_{23}^i s in T_j , we do not want to change one of these D_{23}^i s without changing the others. In other words, we will “tie” these cycles together and change the colors over the entire T_j . But before we do this, we need to ensure we can shift some edges out from between these D_{23}^i s if necessary. Since we have a tree structure between

these D_{23}^i s we can root our tree anywhere and fix an edge-coloring with colors 2 and 3 on that cycle. Then, we travel along edges in the tree, ensuring that along each edge of the tree, at least one edge of M is either a 22- or a 33-edge. We can do this by switching the colors 2 and 3 along each new D_{23}^i encountered. This will allow us to shift this edge to either D_{12} or D_{13} if necessary. After making all of these adjustments to our edges in M , we finally have our starting configuration.

From this point forward, we always think of the edges of G as arranged in a starting configuration. Note that a starting configuration is not unique given a graph G which has a frame made up of the disjoint union of a Kotzig graph and a sturdy graph. This is because the starting configuration depends on which Hamiltonian cycle in H_k we call C_{23} . This is obviously different than the other two Hamiltonian cycles as it is “matched up” with the 2-factor in H_s .

3. LEMMAS

With the above preliminaries in hand, we are finally able to begin proving Theorem 5. We shall present a series of lemmas which narrow down the possibilities of edge types in a graph with a frame made up of a Kotzig and a sturdy graph which does not have a cycle double cover. Throughout, we let G be such a graph which is, again, in a starting configuration.

Lemma 10. *If there is an arrangement of edges of M so that \mathcal{H}_1 , \mathcal{H}_2 and all \mathcal{T}_i s are even, then the graph G has a cycle double cover.*

Proof. In order to prove this lemma, we simply need to show that there is some arrangement of edges in this graph so that each cycle in G has an even number of edges of M incident with it. We may examine each \mathcal{T}_i separately, in any order. So, fix a \mathcal{T}_i . Let the cycles corresponding to vertices of \mathcal{T}_i be $D_{23}^{i_1}, \dots, D_{23}^{i_s}$. Now, since there are an even number of edges in \mathcal{T}_i , we know that there are an even number of odd \mathcal{D}_{i_j} for $j = 1, \dots, s$. Let the set of $D_{23}^{i_j}$ s corresponding to these \mathcal{D}_{i_j} s be denoted S and now apply Corollary 8 to \mathcal{T}_i with S the set of vertices corresponding to the $D_{23}^{i_j}$ s. Then, we know that there is a subforest of \mathcal{T}_i with vertices of odd degree exactly the vertices of S . We then can leave the edge sets of M between cycles of S down, since along with the edges of M in \mathcal{T}_i gives an even number of edges on this cycle. For any two cycles connected by an edge of \mathcal{T}_i not in this set of edge-disjoint paths, we shift an odd number of same colored edges up to the correct Hamiltonian cycle. This gives a configuration of edges of M with each cycle of \mathcal{T}_i incident to an even number of edges of M . Since we can repeat this process with all of the \mathcal{T}_i s, we have proved the lemma. \square

We now prove the following lemma, which states that, in fact, we are done unless we have only one type of edge in either \mathcal{H}_1 or \mathcal{H}_2 in our starting configuration.

Lemma 11. *Any graph which has a frame formed by the disjoint union of a Kotzig graph and a sturdy graph in such a way that it has an edge of the matching between H_s and H_k of color 1 in H_s has a cycle double cover.*

Proof. To begin, note that we may assume that this edge of color 1 in H_s is, in fact, a 11-edge. For, if it is not, and say it is a $i1$ -edge for $i = 2$ or 3 , then switch the colors 1 and i in H_k and we have a new configuration with a 11-edge. Now, we examine the situation amongst the \mathcal{D}_i s and in fact, using Lemma 10, we are interested only in the parity of the \mathcal{T}_i s. We begin by dealing with each \mathcal{T}_i separately, and in arbitrary order. So, begin with \mathcal{T}_1 . If \mathcal{T}_1 is even, then we leave it alone and go to \mathcal{T}_2 . Otherwise, when \mathcal{T}_1 is odd, we examine the types of edges in \mathcal{T}_1 . If there is either a 22- or 33-edge, we simply shift it up to \mathcal{H}_1 or \mathcal{H}_2 . If there is not, then we switch colors 2 and 3 in H_s just amongst cycles in \mathcal{T}_1 . This will create either a 22- or 33-edge, which we shift up to either \mathcal{H}_1 or \mathcal{H}_2 , making our modified \mathcal{T}_1 even. We continue this process for every \mathcal{T}_i , making each

of them even in the process. Thus, when we finish this process, all \mathcal{T}_i s are even, meaning that our new \mathcal{H}_1 and \mathcal{H}_2 are either both odd or both even. If they are both even, we are done. If they are both odd, we may switch the 11-edge (which remains since we didn't switch color 1 in either H_s or H_k) to the other pair of Hamiltonian cycles, making both \mathcal{H}_1 and \mathcal{H}_2 even and completing the proof of the lemma. \square

So, for the remainder of the note, we shall assume that there are no edges of M between H_s and H_k of color 1 in H_s . Thus, in our initial arrangement of edges, we only have edges of type 12 in \mathcal{H}_1 and 13 in \mathcal{H}_2 . Before the next lemma, we define the notion of an *opposite pair of edge types*, which is a pair of edge types of the form either 22 and 33, or 23 and 32.

Lemma 12. *If there exists a \mathcal{T}_i with an opposite pair of edge types appearing, then there exists a cycle double cover of G .*

Proof. Let \mathcal{T}_1 have an opposite pair of edge types inside it. To begin, switch colors 2 and 3 in \mathcal{T}_1 if necessary to make this opposite pair of the form 22 and 33. Then, switch colors 2 and 3 in each of the \mathcal{T}_i s corresponding to the odd \mathcal{T}_i s so that there is an odd number of either 22- or 33-edges in each. Then, for each of these odd \mathcal{T}_i s, shift the odd number of 22- or 33-edges up to either \mathcal{H}_1 or \mathcal{H}_2 , so that each of these \mathcal{T}_i s becomes even. Now, if \mathcal{T}_1 is odd, then one of the \mathcal{H}_i s must be odd and we can simply shift up a 22- or 33-edge from \mathcal{T}_1 to the odd \mathcal{H}_i to get an all even arrangement. If \mathcal{T}_1 is even, then either both \mathcal{H}_i s are even after the shifting from the odd \mathcal{T}_i s and we are done, or they are both odd, in which case we take one 22-edge and one 33-edge from \mathcal{T}_1 , shift them up and we have an all even arrangement. \square

Corollary 13. *If there exists a configuration where any \mathcal{T}_i has more than two types of edges, then there exists a cycle double cover of G .*

Proof. If there is a \mathcal{T}_i with at least three types of edges, then it contains an opposite pair of edge types, and thus we are done by Lemma 12. \square

Before we prove our final lemma, we must define one more object. The key definition in the proof of Theorem 5 follows; it describes a particular type of \mathcal{T}_i which allows us to tweak the parities of the \mathcal{H}_i s after we shift up edges of the form 22 and 33 from the odd \mathcal{T}_i s.

Definition 14. A *controlling \mathcal{T}_i* is an even \mathcal{T}_i with an odd number of edges from one of the \mathcal{H}_i s and an even number from the other in it after switching 1 and either 2 or 3 in H_k .

In other words, we can switch the parities of \mathcal{H}_1 and \mathcal{H}_2 by switching colors 2 and 3 inside the \mathcal{T}_i corresponding to a controlling \mathcal{T}_i . The existence of a controlling \mathcal{T}_i is the key to the rest of the proof, as it implies a cycle double cover of G , as shown in the following lemma.

Lemma 15. *If there exists a starting configuration with a controlling \mathcal{T}_i , then G has a cycle double cover.*

Proof. In this case, we simply shift up all odd types of the form 22 or 33 from each odd \mathcal{T}_i . We then have either both \mathcal{H}_i s even or odd. If they are both even, we are done. If, on the other hand, they are both odd, we simply switch colors 2 and 3 on the \mathcal{T}_i corresponding to the controlling \mathcal{T}_i in the statement of the lemma, and the parities of the \mathcal{H}_i s switch, leaving them both even. \square

We shall next prove the case when there is a starting configuration with an odd \mathcal{T}_i present. Note that there are in fact three starting configurations then for every Kotzig and sturdy graph frame, depending on which Hamiltonian cycle of H_k is chosen to have colors 2 and 3. This starting configuration can be any of the three possible starting configurations.

Lemma 16. *If there is a starting configuration of G with at least one odd \mathcal{T}_i , then G has a cycle double cover.*

Proof. To begin, we note that if color 1 and either 2 or 3 are switched in H_k , then any odd \mathcal{T}_i can only get edges from either \mathcal{H}_1 or \mathcal{H}_2 , but not both. In fact, if it is an odd 22- \mathcal{T}_i , then it can get edges only from \mathcal{H}_1 . Otherwise, we could switch 1 and 3 in H_k , and get a starting configuration in which this \mathcal{T}_i has an opposite pair of edge types, completing the proof by Lemma 12.

Using this fact, we now note that the parity of \mathcal{H}_1 and the number of odd 22- \mathcal{T}_i s must differ, and likewise for \mathcal{H}_2 and odd 33- \mathcal{T}_i s. Otherwise, we can simply shift up all the 22 or 33 edges from these and get an all even configuration. Since there is an odd \mathcal{T}_i , there must be either an odd 22- \mathcal{T}_i or an odd 33- \mathcal{T}_i . Without loss of generality, assume there is an odd 22- \mathcal{T}_i . In fact, we note that since the parity of the number of odd 22- \mathcal{T}_i s is different than \mathcal{H}_1 and also since all odd 22- \mathcal{T}_i s get edges only from \mathcal{H}_1 when colors 1 and either 2 or 3 are switched in H_k , there must be an odd 22- \mathcal{T}_i which gets an even number of edges from \mathcal{H}_1 when colors 1 and 2 or 3 are switched in H_k . Let such a \mathcal{T}_i be \mathcal{T}_j .

To complete the proof, we shall show that we can switch colors in H_k to get a new starting configuration in which \mathcal{T}_j is a controlling \mathcal{T}_i . In fact, we simply switch color 1 and 2 (since we assumed that \mathcal{T}_j has an odd number of 22-edges), and we rearrange to get a new starting configuration. The number of edges in the new \mathcal{T}_j is even, since the number of edges left behind after the switch is even (could be 0) and the number of edges switched down from \mathcal{H}_1 is even (also could be 0). Further, note that the switch puts an even number of edges from this \mathcal{T}_j in our new \mathcal{H}_1 and an even number (perhaps 0) of edges in the new \mathcal{H}_2 . Thus, in this starting configuration, \mathcal{T}_j is a controlling \mathcal{T}_i , and thus we are done after applying Lemma 15. \square

4. PROOF OF THEOREM 5

Finally, we are ready to prove Theorem 5. Essentially the problem that is encountered is that when all the \mathcal{T}_i s are even, there may be several empty \mathcal{T}_i s, to which \mathcal{H}_1 and \mathcal{H}_2 can switch down. This case will have to be dealt with carefully and in a new way.

Proof of Theorem 5. We begin the proof by applying our Lemmas, essentially in order, to obtain a case not yet covered. Thus, Lemma 11 tells us that our frame must not have any edges of color 1 in H_s . Corollary 13 gives that each \mathcal{T}_i can only have two types of edges, and that they must share a color in either H_s or H_k . Lemma 15 gives that there is no starting configuration with a controlling \mathcal{T}_i . Likewise, Lemma 16 gives that every starting configuration must have all even \mathcal{T}_i s. It follows that in each of the three starting configurations, both \mathcal{H}_1 and \mathcal{H}_2 are odd, or we have an all even configuration and are done by Lemma 10.

Now, we consider the case when there is a starting configuration where all \mathcal{T}_i s are nonempty. Since both \mathcal{H}_1 and \mathcal{H}_2 are odd, and by Lemma 15 there are no controlling \mathcal{T}_i s, there must be a \mathcal{T}_i which gets an odd number of edges from both \mathcal{H}_1 and \mathcal{H}_2 . But this \mathcal{T}_i was nonempty and thus had edges with some color in H_k in it. If we switch colors 1 and the other color in H_k , then we get a starting configuration in which this \mathcal{T}_i has three edge types, and thus we are done by Corollary 13.

On the other hand, we may have a situation in which all three starting configurations have empty \mathcal{T}_i s. In this case, we cannot hope to apply Lemma 10 since there are configurations in which we cannot make \mathcal{H}_1 , \mathcal{H}_2 and all \mathcal{T}_i s even, e.g., see Figure 2. In this case, we must have edges between the \mathcal{T}_i s and two of the Hamiltonian cycles in H_k in order to get an even number of edges to each Hamiltonian cycle and \mathcal{T}_i . In this case, there are an odd number of empty \mathcal{T}_i s which have an odd number of edges from each of \mathcal{H}_1 and \mathcal{H}_2 after a switch of color 1 and either 2 or 3 in H_k . We move these edges so that they are on the \mathcal{T}_i in H_s , and since there are an odd number of each, there are an even total number. This gives each \mathcal{T}_i even degree with respect to the matching edges.

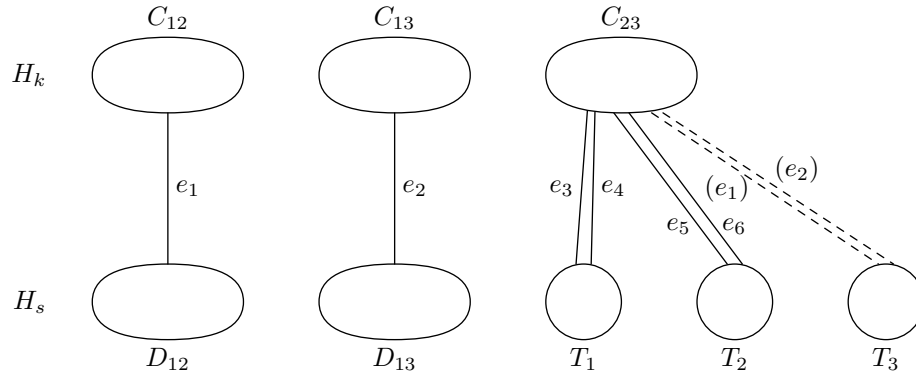


FIGURE 2. An example of a graph with frame H_k and H_s which cannot have an all even configuration with respect to edge sets \mathcal{H}_1 , \mathcal{H}_2 and the \mathcal{T}_i s. Edge e_1 is a 12-edge, e_2 a 13-edge, e_3 a 22-edge, e_4 a 23-edge, e_5 a 32-edge and e_6 a 33-edge. The dotted lines represent where the edges e_1 and e_2 go after switching colors 1 and either 2 or 3 in H_k .

Since all of these edges have color 1, we simply put them on the same Hamiltonian cycle in H_k , and once again there are an even number of them so this gives each Hamiltonian cycle even degree with respect to the matching edges. Now, we must be a bit careful in the end, since we cannot have a cycle which begins in a T_i , goes across matching edges to H_k and then returns to a Hamiltonian cycle of H_s , for this may give a circuit rather than a cycle, i.e., the same vertices may be traversed more than once. However, if we move all edges of the matching down to the T_i s, and have none on D_{12} and D_{13} , then we will certainly avoid this situation. However, since each of the T_i s in this case must have an even number of edges in them not matter which starting configuration is used, we can simply move all matching edges down so that they are on the T_i s in H_s . Since all other sets were even to begin with, we have a configuration in which C_{12} , C_{13} , C_{23} , D_{12} , D_{13} and all T_i s have even degree (where D_{12} and D_{13} have degree 0) with respect to the matching edges. Thus, we have proved the theorem. \square

5. FUTURE DIRECTIONS

The next natural problem to tackle is to prove that the disjoint union of two sturdy graphs is a good frame. Also, considering frames consisting of more than two components would be a step forward. Eventually the aim would be to prove a result which states that all cubic graphs have a frame with less than some number of components, all of which are Kotzig or sturdy. This would, of course, answer the cycle double cover conjecture, so we anticipate quite a bit of work is left to be done.

REFERENCES

- [1] R. Häggkvist and K. Markström, Cycle double covers and spanning minors, I. *J. Combin. Theory Ser. B* **96** (2006), no. 2, 183-206.
- [2] R. Häggkvist and K. Markström, Cycle double covers and spanning minors II. *Discrete Math.* **306** (2006), 762-778.
- [3] F. Jaeger, A survey of the cycle double cover conjecture. In *Cycles in graphs (Burnaby, B.C., 1982)*, pp. 1-12, North-Holland, Amsterdam, 1985.
- [4] P.D. Seymour, Sums of circuits. In *Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977)*, pp. 341-355, Academic Press, New York, 1979.
- [5] G. Szekeres, Polyhedral decompositions of cubic graphs. *Bull. Austral. Math. Soc.* **8** (1973) 367-387.
- [6] C.-Q. Zhang, *Integer flows and cycle covers of graphs*, Marcel Dekker, New York.