MAXIMAL-CLIQUE PARTITIONS AND THE ROLLER COASTER CONJECTURE

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Abstract. A graph G is *well-covered* if every maximal independent set has the same cardinality q. Let $i_k(G)$ denote the number of independent sets of cardinality k in G. Brown, Dilcher, and Nowakowski conjectured that the independence sequence $(i_0(G), i_1(G), \ldots, i_q(G))$ was unimodal for any well-covered graph G with independence number q. Michael and Traves disproved this conjecture. Instead they posited the so-called "Roller Coaster" Conjecture: that the terms

$$i_{\lceil \frac{q}{2} \rceil}(G), i_{\lceil \frac{q}{2} \rceil+1}(G), \ldots, i_q(G)$$

could be in any specified order for some well-covered graph G with independence number q. Michael and Traves proved the conjecture for $q < 8$ and Matchett extended this to $q < 12$.

In this paper, we prove the Roller Coaster Conjecture using a construction of graphs with a property related to that of having a maximal-clique partition. In particular, we show, for all pairs of integers $1 \leq k < q$ and positive integers m, that there is a well-covered graph G with independence number q for which every independent set of size $k+1$ is contained in a unique maximal independent set, but each independent set of size k is contained in at least m distinct maximal independent sets.

1. Introduction

The behavior of the coefficients of the independence polynomial of graphs in various classes has produced many interesting problems. For a graph G, we let $\mathcal{I}(G)$ be the set of independent sets in G, i.e., $\mathcal{I}(G) = \{I \subseteq V(G) : E(G[I]) = \emptyset\}$. Also, let $\mathcal{I}_k(G) = \{I \in \mathcal{I}(G) : |I| = k\}$ and $i_k(G) = |\mathcal{I}_k(G)|$. The *independence number of* G is given by $\alpha(G) = \max\{k \in \mathbb{N} : i_k(G) > 0\}$. We let the *independence polynomial of* G be the polynomial defined by

$$I(G; x) = \sum_{k=0}^{\alpha(G)} i_k(G) x^k.$$

We refer to $(i_0(G), i_1(G), \ldots, i_{\alpha(G)}(G))$ as the *independence sequence of* G.

Natural questions arise when one considers possible orderings of the coefficients of the independence sequence over various classes of graphs. If one considers the class of all graphs, then Alavi, Malde, Schwenk, and Erdős [1] proved that the coefficients can be ordered in any way apart from $i_0(G) = 1$. In particular, they proved the following. Throughout the paper, we let $[n] = \{1, 2, \ldots, n\}$.

Theorem 1.1 (Alavi, Malde, Schwenk, Erdős [1]). *Given a positive integer q and a permutation π of $[q]$, there is a graph G with $\alpha(G) = q$ such that*

$$i_{\pi(1)}(G) < i_{\pi(2)}(G) < \cdots < i_{\pi(q)}(G).$$

A graph G is said to be *well-covered* if every maximal independent set in G has the same size. Brown, Dilcher, and Nowakowski [5] conjectured that the independence sequence of any well-covered graph is unimodal. This conjecture was disproved by Michael and Traves [8]. However, they were able to show the following.

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Theorem 1.2 (Michael, Traves [8]). *The independence sequence of a well-covered graph G with $\alpha(G) = q$ satisfies*

$$\frac{i_0(G)}{\binom{q}{0}} \leq \frac{i_1(G)}{\binom{q}{1}} \leq \ldots \leq \frac{i_q(G)}{\binom{q}{q}}.$$

This implies the following.

Corollary 1.3 (Michael, Traves [8]). *If G is a well-covered graph with $\alpha(G) = q$, then*

$$i_0(G) < i_1(G) < \cdots < i_{\lceil q/2 \rceil}(G).$$

In addition, Michael and Traves conjectured that the second half of the independence sequence can be "any-ordered". To be precise, they conjectured the following, which has become known as the Roller Coaster Conjecture.

Conjecture 1.4 (Michael, Traves [8]; Roller Coaster Conjecture). *Given a positive integer q and a permutation π of $\{\lceil q/2 \rceil, \lceil q/2 \rceil + 1, \ldots, q\}$, there is a well-covered graph G with $\alpha(G) = q$ and*

$$i_{\pi(\lceil q/2 \rceil)}(G) < i_{\pi(\lceil q/2 \rceil + 1)}(G) < \cdots < i_{\pi(q)}(G).$$

In addition, Michael and Traves proved the conjecture for $q \leq 7$. Matchett [7] was able to prove the Roller Coaster Conjecture for $q \leq 11$. He also proved that for sufficiently large q, the last $(.1705)q$ terms in the independence sequence of some well-covered graph can be any-ordered. Related work has been done in the context of pure O-sequences in order ideals [4].

We will show that a partial converse to Theorem 1.2 is true. Consider the following definition.

Definition 1. We say that a polynomial $a_q x^q + \cdots + a_1 x$ is an *approximate well-covered independence polynomial* if for every $\epsilon > 0$, there exists a well-covered graph G of independence number q and a real number T such that for all $1 \leq k \leq q$,

$$\left| \frac{i_k(G)}{T} - a_k \right| < \epsilon. \tag{1}$$

Given such real numbers T, ϵ and graph G, say that G is an *ϵ-certificate for $a_q x^q + \ldots + a_1 x$ with scaling factor T.*

Theorem 1.2 implies that for an approximate well-covered independence polynomial $a_q x^q + \ldots + a_1 x$, we have

$$\frac{a_1}{\binom{q}{1}} \leq \frac{a_2}{\binom{q}{2}} \leq \ldots \leq \frac{a_q}{\binom{q}{q}}. \tag{2}$$

We will show that given a sequence of non-negative real numbers $(a_1, a_2, \ldots, a_q)$ satisfying (2), the polynomial $\sum_{i=1}^{q} a_i x^i$ is an approximate well-covered independence polynomial. In order to do this, we will construct well-covered graphs with independence sequence satisfying (1) for some real number T. We will construct these graphs from graphs satisfying the following property.

Definition 2. For integers $0 \leq k < q$ and $1 \leq m$, say that graph G satisfies the property $P(k, q; m)$ if:

(1) All maximal cliques in G are of size q,
(2) Each clique of size $k + 1$ in G is contained in a unique maximal clique, and
(3) Each clique of size k in G is contained in at least m maximal cliques.

Note that if G satisfies property $P(k, q; m)$, then its complement is a well-covered graph with independence number q. It seems that graphs that satisfy property $P(k, q; m)$ have not been studied up to this point, but they are related to the study of maximal-clique covers and partitions in graphs (see, e.g., [9]). A *maximal-clique covering* of a graph G is a set of maximal cliques in G whose union contains each edge of G at least once. A maximal-clique covering in which every edge is in exactly one element of the covering is a *maximal-clique partition.* In our case, instead of covering edges, we are covering cliques of size $k+1$ with maximal cliques. In addition to this, we are covering cliques of size k with at least m distinct maximal cliques. Clique coverings have recently been found to have implications in design theory (see, e.g., [2]) and so graphs satisfying $P(k,q;m)$ may as well.

In Section 2 we give a construction of graphs which satisfy property $P(k,q;m)$. In Section 3, we use these graphs to prove that (2) is a sufficient condition for $a_q x^q + \cdots + a_1 x$ to be an approximate well-covered independence polynomial. Finally, in Section 4, we show that this implies the Roller Coaster Conjecture, i.e., we prove the following.

Theorem 1.5. *Given a positive integer q and a permutation π of $\{\lceil q/2\rceil, \lceil q/2\rceil+1, \ldots, q\}$, there is a well-covered graph G with $\alpha(G) = q$ and*

$$i_{\pi(\lceil q/2\rceil)}(G) < i_{\pi(\lceil q/2\rceil+1)}(G) < \cdots < i_{\pi(q)}(G).$$

2. Graph Construction

For a set S and positive integer k, let $\binom{S}{k} = \{A \subseteq S : |S| = k\}$. Fix integers k, q, and m with $1 \le k < q$ and $1 \le m$. For $i \in [q]$, define $\mathcal{F}_i^{k,q;m}$ to be the following set of functions:

$$\mathcal{F}_i^{k,q;m} = \left\{ f : \binom{[q]\setminus\{i\}}{k} \to [m] \right\}.$$

Our graph is defined in terms of elements of $\mathcal{F}_i^{k,q;m}$.

Definition 3. For integers k, q, and m with $1 \le k < q$ and $1 \le m$, we define $H_{k,q;m}$ to be the graph with vertex set

$$\bigcup_{i=1}^{q} \mathcal{F}_i^{k,q;m},$$

and, for $f \in \mathcal{F}_i^{k,q;m}$ and $g \in \mathcal{F}_j^{k,q;m}$, we let $f \sim g$ if and only if $i \ne j$ and

$$f\big|_{\mathcal{A}} = g\big|_{\mathcal{A}},$$

where $\mathcal{A} = \binom{[q]\setminus\{i,j\}}{k}$. If $k=0$, we define $H_{0,q;m}$ to be the disjoint union of m copies of K_q.

For example, if $m = 1$ and $k \ge 1$, then $\mathcal{F}_i^{k,q;m}$ consists of one (constant) function and so $H_{k,q;m} = K_q$. See Figure 1 for a depiction of the graph $H_{1,3;2}$.

For a function $f : \binom{[q]}{k} \to [m]$, denote by C_f the set of restrictions of f to $\binom{[q]\setminus\{i\}}{k}$ for $1 \le i \le q$. Note that C_f has size q.

Lemma 2.1. *For integers k, q, and m with $1 \le k < q$ and $1 \le m$, every clique in $H_{k,q;m}$ is contained in a clique of the form C_f for some function $f : \binom{[q]}{k} \to [m]$, and so each maximal clique in $H_{k,q;m}$ is of size q. Furthermore, each clique of size $k+1$ is contained in a unique such clique, while every clique of size k is contained in m distinct such cliques.*

Proof. In order for a set of vertices in $H_{k,q;m}$, say $\{f_1, f_2, \ldots, f_r\}$, to be a clique, it must be the case that, for each $j \in [r]$, there is an $i_j \in [q]$ such that $f_j \in \mathcal{F}_{i_j}^{k,q;m}$. Further, we have $i_j \ne i_k$ if $j \ne k$. If $A \in \binom{[q]}{k}$ and $i_j \notin A$ for some $j \in [r]$, then A is in the domain of f_j and any other function in the clique must agree with f_j on A (provided A is in its domain). Thus, any clique in $H_{k,q;m}$ consists of restrictions of functions of the form $f : \binom{[q]}{k} \to [m]$. Note that if $B \in \binom{[q]}{k}$ and $B \supseteq \{i_j : j \in [r]\}$, then B is not in the domain of any of the functions in the clique.

Consider a clique in $H_{k,q;m}$ of size $k+1$ consisting of vertices $\{g_1, g_2, \ldots, g_{k+1}\}$ where $g_j \in \mathcal{F}_{i_j}^{k,q;m}$ for $j \in [k+1]$. There is no $A \in \binom{[q]}{k}$ such that $A \supseteq \{i_j : j \in [k+1]\}$. Thus, there is a unique $f : \binom{[q]}{k} \to [m]$ such that $g_j \in C_f$ for each $j \in [k+1]$ and so there is a unique q-clique containing the $(k+1)$-clique.

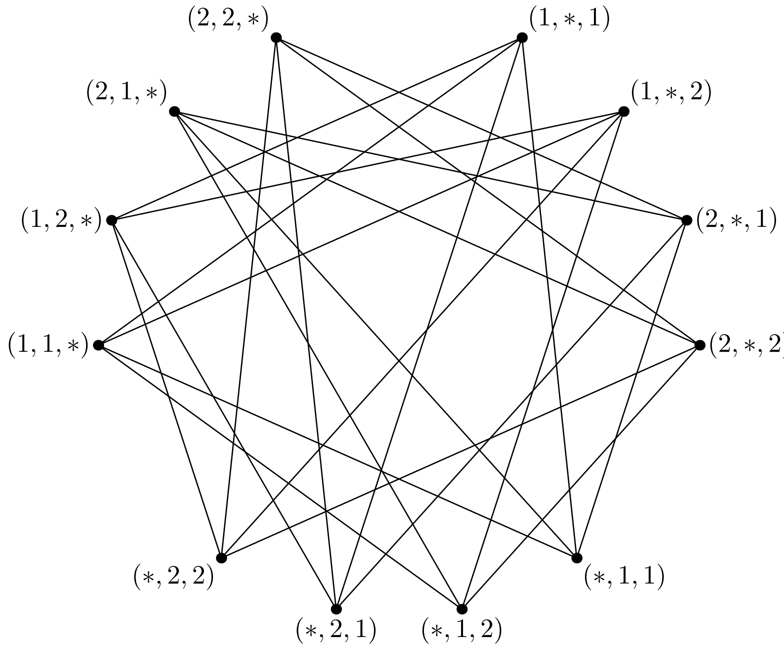

FIGURE 1. The graph $H_{1,3;2}$. The vertices are labeled with the images of 1, 2, and 3, respectively, where the $*$ represents the element of $[3]$ excluded from the domain of the function. Note that each vertex is in two triangles, but each edge is in a unique triangle.

If $\{h_1, h_2, \ldots, h_k\}$ is a k-clique in $H_{k,q;m}$ where $h_j \in \mathcal{F}_{i_j}^{k,q;m}$ for $j \in [k]$, then $B = \{i_j : j \in [k]\} \in \binom{[q]}{k}$ is not in the domain of any of the h_js. Since a value on B has not been specified, there are m functions $f : \binom{[q]}{k} \to [m]$ such that $h_j \in C_f$ for all $j \in [k]$. Therefore, the k-clique is contained in at least m maximal cliques in $H_{k,q;m}$. □

Theorem 2.2. *For any integers k, q, and m with $0 \leq k < q$ and $1 \leq m$, the graph $H_{k,q;m}$ satisfies property $P(k, q; m)$.*

Proof. The case when $k \geq 1$ immediately from Lemma 2.1. When $k = 0$, we have $H_{0,q;m} = mK_q$. Each vertex, or K_1, in mK_q is in a unique K_q, while the empty set is in all m of the K_qs. Thus, $H_{0,q;m}$ satisfies $P(0, q; m)$. □

3. Partial converse to Theorem 1.2

Our main goal in this section is to prove the following theorem.

Theorem 3.1. *For all positive integers q and sequences of real numbers $(a_1, \ldots, a_q)$ satisfying*

$$\frac{a_1}{\binom{q}{1}} \leq \frac{a_2}{\binom{q}{2}} \leq \ldots \leq \frac{a_q}{\binom{q}{q}},$$

$a_1 x + a_2 x^2 + \ldots + a_q x^q$ is an approximate well-covered independence polynomial.

We begin by using the graphs $H_{k,q;m}$ to generate approximate well-covered independence polynomials.

Lemma 3.2. *For all integers $0 \leq k < q$, the polynomial $\sum_{j=k+1}^{q} \binom{q}{j} x^j$ is an approximate well-covered independence polynomial.*

Proof. Fix $\epsilon > 0$, and let m be a positive integer such that $\frac{2^q}{m} < \epsilon$. Note that, by Theorem 2.2, for integers k, q, and m with $0 \leq k < q$ and $1 \leq m$, $H_{k,q;m}$ satisfies property $P(k,q;m)$ and so the complement of $H_{k,q;m}$ is a well-covered graph with independence number q.

Suppose $H_{k,q;m}$ has T cliques of size q, and let us consider the number of cliques of size j for $1 \leq j \leq q$. Clearly, there are $T\binom{q}{j}$ pairs of cliques (K_1, K_2) such that K_1 is of size q, K_2 of size j, and $K_2 \subseteq K_1$. If $j \geq k+1$, then each clique of size j contains a clique of size $k+1$, and hence is contained in at most one clique of size q. Since all maximal cliques are of size q, each clique of size j is contained in a unique clique of size q, and hence there are $T\binom{q}{j}$ cliques of size j. On the other hand, if $1 \leq j \leq k$, then each clique of size j is contained in a clique of size k and is therefore contained in at least m cliques of size q. Hence there are at most $T\binom{q}{j}/m < T\epsilon$ cliques of size j.

Thus, if G is the complement of $H_{k,q;m}$, then $\frac{i_j(G)}{T} = \binom{n}{j}$ for $j \geq k+1$. For $1 \leq j \leq k$, we have $\left|\frac{i_j(G)}{T} - 0\right| < \epsilon$, so G is an ϵ-certificate for $\sum_{j=k+1}^{n}\binom{n}{j}x^j$ with scaling factor T. □

Now we show that the class of approximate well-covered independence polynomials of a given degree is additive. In order to do this, we use the join[1] operation on graphs. Note that if G and H are graphs and $k \geq 1$, then $i_k(G \vee H) = i_k(G) + i_k(H)$.

Lemma 3.3. *If $P_1(x)$ and $P_2(x)$ are approximate well-covered independence polynomials of degree q, then $P_1(x) + P_2(x)$ is an approximate well-covered independence polynomial of degree q.*

Proof. Fix $\epsilon > 0$. Let G_1, G_2 be $\frac{\epsilon}{3}$-certificates of $P_1(x)$ and $P_2(x)$ with scaling factors T_1 and T_2 respectively. Suppose that all coefficients of $P_1(x)$ and $P_2(x)$ are bounded above by N, and let k_1, k_2 be positive integers such that

$$1 - \frac{\min(k_1T_1, k_2T_2)}{\max(k_1T_1, k_2T_2)} < \frac{\epsilon}{6N}.$$

Define $T := \max(k_1T_1, k_2T_2)$.

Let G be the graph defined as the join of k_1 copies of G_1 joined to the join of k_2 copies of G_2, i.e.,

$$G = \left(\bigvee_{i=1}^{k_1} G_1\right) \vee \left(\bigvee_{i=1}^{k_2} G_2\right).$$

All independent sets in G are completely contained in a single copy of G_1 or a single copy of G_2. As such, G is well-covered (since $q(G_1) = q(G_2) = q$) and $i_j(G) = k_1 i_j(G_1) + k_2 i_j(G_2)$ for all $j \geq 1$.

Suppose that, for $j \geq 1$, the x^j coefficients of $P_1(x)$ and $P_2(x)$ are $p_j^{(1)}$ and $p_j^{(2)}$, respectively. Then since G_1 is a $\frac{\epsilon}{3}$-certificate of $P_1(x)$ with scaling factor T_1, by definition, $\left|p_j^{(1)} - \frac{i_j(G_1)}{T_1}\right| < \frac{\epsilon}{3}$. Thus, since $k_1T_1 \leq T$, we have that

$$\left|\frac{k_1T_1p_j^{(1)}}{T} - \frac{k_1 i_j(G_1)}{T}\right| < \frac{\epsilon}{3}.$$

Further, $p_j^{(1)} < N$ and

$$1 - \frac{k_1T_1}{T} < 1 - \frac{\min(k_1T_1, k_2T_2)}{\max(k_1T_1, k_2T_2)} < \frac{\epsilon}{6N},$$

[1]The *join* of graphs G and H, denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

and so we have $\left|p_j^{(1)} - \frac{k_1 T_1 p_j^{(1)}}{T}\right| < \frac{\epsilon}{6}$. Thus, we see that

$$\left|p_j^{(1)} - \frac{k_1 i_j(G_1)}{T}\right| < \frac{\epsilon}{2}.$$

Similarly,

$$\left|p_j^{(2)} - \frac{k_2 i_j(G_2)}{T}\right| < \frac{\epsilon}{2}.$$

It follows that

$$\left|p_j^{(1)} + p_j^{(2)} - \frac{k_1 i_j(G_1) + k_2 i_j(G_2)}{T}\right| = \left|p_j^{(1)} + p_j^{(2)} - \frac{i_j(G)}{T}\right| < \epsilon,$$

so G is an ϵ-certificate of $P_1(x) + P_2(x)$ with scaling factor T. □

The same is true for more complicated linear combinations.

Lemma 3.4. *If $P_1(x), P_2(x), \ldots, P_k(x)$ are approximate well-covered independence polynomials of degree q, and $\lambda_1, \ldots, \lambda_k$ are positive real numbers then $\sum_{i=1}^k \lambda_i P_i(x)$ is an approximate well-covered independence polynomial of degree q.*

Proof. If G is an ϵ-certificate of $P_i(G)$ with scaling factor T, then it is a $\lambda_i\epsilon$-certificate of $\lambda_i P_i(G)$ with scaling factor $\frac{T}{\lambda_i}$. Thus for each $i \in [k]$, $\lambda_i P_i(G)$ is an approximate well-covered independence polynomial of degree q, and therefore the sum of them is by Lemma 3.3. □

With these lemmas in hand, we are now ready to prove the main result of this section, i.e., Theorem 3.1.

Proof of Theorem 3.1. Fix a sequence $a_1, a_2, \ldots, a_q$ satisfying

$$\frac{a_1}{\binom{q}{1}} \leq \frac{a_2}{\binom{q}{2}} \leq \ldots \leq \frac{a_q}{\binom{q}{q}}.$$

Let $b_1 = \frac{a_1}{\binom{q}{1}}$ and, for $i > 1$, let $b_i = \frac{a_i}{\binom{q}{i}} - \frac{a_{i-1}}{\binom{q}{i-1}}$. Then, for all i, $b_i > 0$ and

$$a_i = \binom{q}{i} \sum_{j=1}^{i} b_j.$$

Let $P_k(x) = \sum_{j=k}^q \binom{q}{j} x^j$ so, by Lemma 3.2, $P_k(x)$ is an approximate well-covered independence polynomial for all $1 \leq k \leq q$. Therefore, by Lemma 3.4, so is

$$\sum_{j=1}^{q} b_j P_j(x) = \sum_{j=1}^{q} b_j \sum_{i=j}^{q} \binom{q}{i} x^i = \sum_{i=1}^{q} \left(\sum_{j=1}^{i} \binom{q}{j} b_j \right) x^i = \sum_{i=1}^{q} a_i x^i. \qquad \square$$

4. Proof of the Roller Coaster Conjecture

Finally, we will show that Theorem 3.1 itself implies the Roller Coaster Conjecture.

Lemma 4.1. *If $a_q x^q + \ldots + a_1 x$ is an approximate well-covered independence polynomial and S is a subset of $[q]$ such that $a_i \neq a_j$ if $i \neq j$ and $i, j \in S$, then there exists a well-covered graph G of independence number q such that for all $j, k \in S$, $i_j(G) < i_k(G)$ if and only if $a_j < a_k$.*

Proof. Let

$$\epsilon = \frac{1}{3} \min \{|a_i - a_j| \, : \, i \neq j, i, j \in S\}.$$

Note that $\epsilon > 0$. Let G be an ϵ-certificate of $a_q x^q + \ldots + a_1 x + a_0$ with scaling factor T. Then G is a well-covered graph of independence number q such that for all j, $|\frac{i_j(G)}{T} - a_j| < \epsilon$.

For $j, k \in S$, if $a_j < a_k$, we have $3\epsilon \leq a_k - a_j$. Thus, $a_j + \epsilon < a_k - \epsilon$, and so

$$i_j(G) < T(a_j + \epsilon) < T(a_k - \epsilon) < i_k(G). \qquad \square$$

Therefore, if we can any-order the initial coefficients of approximate well-covered independence polynomials, we can do the same for actual well-covered independence polynomials.

Lemma 4.2. *For any integer n and for any permutation π of the set $\{\lceil q/2\rceil, \lceil q/2\rceil + 1, \ldots, q\}$, there exists an approximate well-covered independence polynomial $a_q x^q + \cdots + a_1 x + a_0$ such that for all $\lceil \frac{q}{2}\rceil \leq k, l \leq q$, $a_k < a_l$ if and only if $\pi(k) < \pi(l)$.*

Proof. Define the sequence $(a_1, a_2, \ldots, a_q)$ as follows.

$$a_i = \begin{cases} \binom{q}{i} & \text{if } 1 \leq i < \lceil \frac{q}{2} \rceil, \\ 2^q + \pi(i) & \text{if } \lceil \frac{q}{2} \rceil \leq i \leq q. \end{cases}$$

Then $\frac{a_i}{\binom{q}{i}} = 1$ for $1 \leq i < \lceil \frac{q}{2}\rceil$, while $\frac{a_i}{\binom{q}{i}} > 1$ for $\lceil \frac{q}{2}\rceil \leq i$. Further, for $\lceil \frac{q}{2}\rceil \leq i < q$,

$$\frac{a_i}{a_{i+1}} \leq \frac{2^q + q}{2^q} \leq 1 + \frac{2}{q},$$

while

$$\frac{\binom{q}{i}}{\binom{q}{i+1}} = \frac{i+1}{q-i} \geq \frac{\frac{q}{2}+1}{\frac{q}{2}} = 1 + \frac{2}{q}.$$

It follows that

$$\frac{a_1}{\binom{q}{1}} \leq \frac{a_2}{\binom{q}{2}} \leq \ldots \leq \frac{a_q}{\binom{q}{q}}.$$

Therefore, by Theorem 3.1, $a_q x^q + \ldots + a_1 x$ is an approximate well-covered independence polynomial. Furthermore, for $\lceil \frac{q}{2}\rceil \leq k, l \leq q$, $a_k = 2^q + \pi(k) < a_l = 2^q + \pi(l)$ if and only if $\pi(k) < \pi(l)$. $\square$

Our main theorem, Theorem 1.5, follows.

Proof of Theorem 1.5. The statement follows from applying Lemma 4.2 and then Lemma 4.1 with $S = \{\lceil q/2\rceil, \lceil q/2\rceil + 1, \ldots, q\}$. $\square$

5. Conclusion

Many interesting questions about the independence sequence of graphs are still open. It was conjectured by Levit and Mandrescu [6] that every König-Egerváry graph (a graph G with $\alpha(G) + \nu(G) = n(G)$, where $\nu(G)$ is the size of the largest matching in G and $n(G)$ is the number of vertices in G) has a unimodal independence sequence. This conjecture was recently disproved by Bhattacharyya and Kahn [3], who provided a bipartite graph with non-unimodal independence sequence (since every bipartite graph is a König-Egerváry graph). However, the following conjecture of Alavi et al. is still open.

Conjecture 5.1 (Alavi, Malde, Schwenk, Erdős [1])**.** *Every tree and forest has unimodal independence sequence.*

We also believe that graphs satisfying property $P(k, q; m)$ may be of independent interest. Often the question for such structures is how small can such an object be? To be precise, our question is as follows.

Question. Given integers k, q, and m with $0 \leq k < q$ and $m \geq 1$, what is the minimum number of vertices in a graph G with property $P(k, q; m)$?

The graph $H_{k,q;m}$ has $qm^{\binom{q-1}{k}}$ vertices which we suspect is far from the minimum. Recall that, for integers k and n with $1 \leq k \leq n$, the Kneser graph $KG_{n,k}$ is the graph with vertex set $\binom{[n]}{k}$, where two vertices are adjacent if and only if they are disjoint. One can check that $KG_{q(q-2),q-2}$ satisfies property $P(q-2, q; \frac{1}{2}\binom{2(q-2)}{q-2})$. Further, we have

$$n(H_{q-2,q;m}) = qm^{q-1} \gg \binom{q(q-2)}{q-2} = n(KG_{q(q-2),q-2})$$

when $m = \frac{1}{2}\binom{2(q-2)}{q-2}$.

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