UNAVOIDABLE SUBGRAPHS OF COLORED GRAPHS

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Abstract. A natural generalization of graph Ramsey theory is the study of unavoidable subgraphs in large colored graphs. In this paper, we find a minimal family of unavoidable graphs in two-edge-colored graphs. Namely, for a positive even integer k, let $\mathcal{S}_k$ be the family of two-edge-colored graphs on k vertices such that one of the colors forms either two disjoint $K_{k/2}$s or simply one $K_{k/2}$. Bollobás conjectured that for all k and $\varepsilon > 0$, there exists an $n(k,\varepsilon)$ such that if $n \geq n(k,\varepsilon)$ then every two-edge-colorings of K_n, in which the density of each color is at least ε, contain a member of this family. We solve this conjecture and present a series of results bounding $n(k,\varepsilon)$ for different ranges of ε. In particular, if ε is sufficiently close to $1/2$, the gap between our upper and lower bounds for $n(k,\varepsilon)$ is smaller than those for the classical Ramsey number $R(k,k)$.

1. Introduction

Ramsey theory has been the subject of many papers by many mathematicians in many different fields. While classical Ramsey theory is the study of monochromatic substructures and anti-Ramsey theory is the study of totally multicolored substructures, there are many other colorings that can be studied. In this paper we shall study specifically colored substructures somewhere between the goals of the above fields; our main aim is to solve a problem of Bollobás.

Before we begin to describe the aims of this paper, let us recall some basic facts of Ramsey theory. The foundations of the theory go back to papers of Ramsey [8] and Erdős and Szekeres [5]. In particular, Erdős and Szekeres gave the following lower bound for the maximal order of a monochromatic subgraph in a 2-edge-colored complete graph. Let $R(k,k)$ be the minimal value of n such that every 2-coloring of the edges of K_n contains a monochromatic K_k.

Theorem 1. *For $k \geq 1$, $R(k,k) < 4^k/\sqrt{k}$.*

Over the years, the bound in Theorem 1 has been improved only a little, with the best bound proved by Thomason [11].

Theorem 2. *There is an absolute constant A such that*

$$R(k,k) < \frac{4^k}{k^{1-A/\sqrt{\log k}}}.$$

In his seminal paper, introducing the probabilistic method, Erdős [3] provided a lower bound for $R(k,k)$.

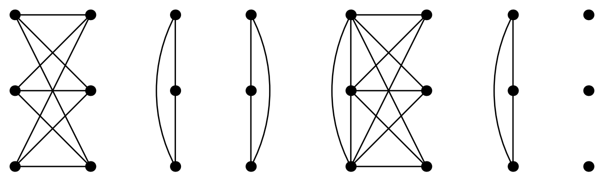
Theorem 3. *For $k \to \infty$, we have $R(k,k) > (1/e\sqrt{2} + o(1))k2^{k/2}$.*

By a simple application of the Lovász Local Lemma [4], Spencer [9] improved this bound by a factor of 2.

Theorem 4. *For $k \to \infty$, we have $R(k,k) > (\sqrt{2}/e + o(1))k2^{k/2}$.*

In this paper, we shall not be interested in finding monochromatic complete subgraphs, but rather other specifically colored subgraphs. Since no such graphs will appear if K_n is monochromatic, we need some condition on the 2-coloring of K_n. Among the many possibilities, perhaps the most natural is to prescribe the density of both colors to be close to 1/2. Also, this prescription is not all that restrictive; for if we allow the density of either color to differ from 1/2 by at most $\Omega(1/n)$, then we still consider almost every 2-coloring. Ramsey questions involving 2-edge-colored graphs with both colors having positive density are not new; see, for example, Székely [10].

As alluded to above, we would like to be able to find a family of graphs which acts as the monochromatic complete graphs in the general case, i.e., a family which is unavoidable under our density condition. We consider the following 2-colored subgraphs: if k is a positive even integer, let $\mathcal{S}_k$ be the set of 2-edge-colored graphs on k vertices in which one color class forms either a $K_{k/2}$ or two vertex-disjoint $K_{k/2}$'s. In other words, if we color a K_k red and blue, then it can belong to $\mathcal{S}_k$ in four different ways: the monochromatic cut between the two monochromatic $K_{k/2}$'s can be red or blue, and in both cases the other color can form one or two $K_{k/2}$'s. See Figure 1 for an illustration of the elements of $\mathcal{S}_6$.



I. II. III. IV.

FIGURE 1. The members of $\mathcal{S}_6$, with edges being the elements of one color class and non-edges elements of the other.

We shall now show that if, in fact, the members of $\mathcal{S}_k$ are unavoidable, then they form a minimal such family. In other words, even with our density condition, no three of these four colored subgraph's existence can be guaranteed for any n. Indeed, it is easy to see that if we leave out the coloring in which a color class (say, red) forms two $K_{k/2}$'s, then the other three colored subgraphs belonging $\mathcal{S}_k$ are avoided by coloring the edges of two vertex-disjoint cliques with order $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ red, and all edges connecting them blue. Similarly, if we leave out the coloring in which a color class (say again, red) forms one $K_{k/2}$, then the other three colored subgraphs are avoided by coloring the edges of a $K_{\lfloor n/\sqrt{2} \rfloor}$ red, and all of the remaining edges blue.

Once again, the aim of this paper is to show that $\mathcal{S}_k$ behaves in a very similar way as the family of monochromatic K_ks. More precisely, we shall prove a conjecture of Bollobás [1], that, for any positive ε, there is an n such that every 2-coloring of the edges of K_n with the density of both colors at least ε yields a member of $\mathcal{S}_k$. Furthermore, if both densities are at least $1/2 - O(1/k)$, then we shall estimate the smallest appropriate n with the same accuracy as we currently can estimate the classical Ramsey number $R(k,k)$. In fact, the main interest of the paper is the accuracy with which the upper bound can be estimated. Thus, a substantial amount of the paper is made up of rather detailed calculations. Before we start to discuss the considerably more interesting upper bound on this n, we outline the proof of a lower bound which follows fairly easily from an appropriate modification of the original proof of Theorem 4. Our lower bound is essentially half of the bound which appears in that theorem, even in the case when the number of the red edges is exactly $\lfloor \binom{n}{2}/2 \rfloor$. Thus we shall show that, if $n < (1/e\sqrt{2} + o(1))k2^{k/2}$, then there exists a 2-edge-coloring of K_n with $\lfloor \binom{n}{2}/2 \rfloor$ red edges that does not yield any member of $\mathcal{S}_k$.

The original proof of Theorem 4 considers a random 2-edge-coloring of K_n, and the probability p of the event that, in a uniformly chosen random 2-edge-coloring of K_n, the subclique induced by given k vertices a given subclique on k vertices is colored appropriately. Spencer showed, applying

Lovász Local Lemma, that $4\binom{k}{2}\binom{n}{k-2}p < 1$ is sufficient for the existence of a 2-edge-coloring of K_n that does not yield any appropriately colored K_k. In the classical case, when the appropriate colorings are monochromatic and there is no condition on the densities, $p = 2^{1-\binom{k}{2}}$, since the probability of *every* given 2-edge-coloring on a labelled K_k is $2^{-\binom{k}{2}}$, and there are two appropriate colorings.

We begin by restricting our probability space of 2-edge-colorings of K_n to only those colorings in which the number of red edges is $\lfloor\binom{n}{2}/2\rfloor$. Under this restriction, the probabilities of different 2-edge-colorings on a K_ks occurring in our K_n are different. In order to get around this problem, we show that there is a constant c such that if $n > ck^2$ then, for two probabilities p_1, p_2 of any two colorings, $p_1/p_2 \leq 2$. If the corresponding two colorings have r_1, r_2 red edges, respectively, then

$$p_1/p_2 = \binom{\binom{n}{2}-\binom{k}{2}}{\lfloor\binom{n}{2}/2\rfloor - r_1} \Big/ \binom{\binom{n}{2}-\binom{k}{2}}{\lfloor\binom{n}{2}/2\rfloor - r_2} \leq \binom{\binom{n}{2}-\binom{k}{2}}{\frac{\binom{n}{2}-\binom{k}{2}}{2}} \Big/ \binom{\binom{n}{2}-\binom{k}{2}}{\binom{n}{2}/2},$$

and the right hand side is at most 2 for $n > ck^2$ with an appropriate c.

This implies that, since there are $2^{\binom{k}{2}}$ 2-edge-colorings of a given K_k, the probability of any of them is at most $2^{1-\binom{k}{2}}$. It is easy to see that there are $3\binom{k}{k/2}$ ways to color a K_k in order for it to be a member of $\mathcal{S}_k$. Therefore, if $4\binom{k}{2}\binom{n}{k-2}3\binom{k}{k/2}2^{1-\binom{k}{2}} < 1$, then the existence of a 2-edge-coloring of K_n that does not yield any member of $\mathcal{S}_k$ is guaranteed. An application of Stirling's formula shows that $n < (1/e\sqrt{2}+o(1))k2^{k/2}$ is sufficient for this condition to hold. We do not think that an improvement of this statement, even without any density condition, is any easier than an improvement of Theorem 4.

For the upper bound of the smallest appropriate n, however, one can not use the classical methods of bounding $R(k,k)$. Through different techniques, then, we are able to show the following result.

Theorem 5. *There are absolute constants C and k_0 with the following property. Let n, k, m be positive integers with k even such that $k > k_0$. Also, let ϱ and d be such that $1/30 \leq \varrho < 16/30$ and $d \geq \frac{k-2}{2m} + \frac{\varrho k}{m^2}$. Suppose that a 2-edge-coloring of K_n with density of each color at least d does not yield any member of $\mathcal{S}_k$. Then*

$$n < \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{m}}\right)\frac{4^m}{m^{1-A/\sqrt{\log m}}},$$

where A is the constant appears in Theorem 2.

Using Theorem 5, we are able to show a series of results, each of which prescribes different conditions of the minimal density of the colors used in the 2-edge-colored K_n. The first gives a general result showing the existence of an n guaranteeing a member of $\mathcal{S}_k$ in a 2-edge-colored K_n for any positive density of the colors.

Corollary 6. *Let $d(k)$ be an arbitrary function mapping positive even integers into positive real numbers. Then for every positive even integer k there is an integer $n(k)$ such that if a 2-edge-coloring of K_n with density of each color at least $d(k)$ does not yield any member of $\mathcal{S}_k$, then $n < n(k)$.*

Our second corollary gives a bound on n in terms of the density of each color in a more transparent way than in Theorem 5. Also, note that in this case the density does not depend on k.

Corollary 7. *There is an absolute constant D with the following property: Let d be real with $0 < d < 1/2$ and n, k be positive integers, with k even. Then if a 2-edge-coloring of K_n with density* *of each color at least d does not yield any member of $\mathcal{S}_k$,*

$$n < \left(d4^{-1/d+47/30} + \frac{D}{\sqrt{\log k}}\right)\frac{4^{k/2d}}{k^{1-A/\sqrt{\log k}}},$$

where A is the constant appears in Theorem 2.

Finally, our last corollary makes the relationship between our result and classical Ramsey theorems clear, as is described after its statement.

Corollary 8. *Let $c \geq 0$ be a real number, and q, r the unique pair such that q is a positive integer, $1/30 < r \leq 16/30$ and $c = q/2 - r$. Then there is a real number E with the following property: Let n, k be positive integers, with k even such that $k > 2c$. Then if a 2-edge-coloring of K_n with density of each color at least $1/2 - c/k$ does not yield any member of $\mathcal{S}_k$,*

$$n < \frac{4^{q-1}}{\sqrt{30r}}\left(1 + \frac{E}{\sqrt{k}}\right)\frac{4^k}{k^{1-A/\sqrt{\log k}}},$$

where A is the constant appears in Theorem 2.

We note that if $c = 0$ then $q = 1$ and $r = 1/2$, and hence our bound on n is asymptotically the current upper bound on $R(k,k)$ divided by $\sqrt{15}$. The bound is stronger than the bound on $R(k,k)$ when $q = 1$ and $r > 2/15$, that is, for $c < 11/30$. In other words, if we prescribe densities of each color at least $1/2 - 11/30k$, then the gap between our lower and upper bounds on the smallest appropriate n is slightly narrower than the gap on the current bounds on $R(k,k)$. Our corollary also claims a bound on n asymptotically equal to the bound on $R(k,k)$ for $c = 7/15$.

The next section will provide the lemmas required in the proof of the theorem, including two results related to a classical theorem of Kővári, T. Sós and Turán theorem [7]. The following section includes the proof of the main theorem, while the last section consists of proofs of its corollaries.

2. The lemmas

In this section we shall present the lemmas used in the proof of Theorem 5. We begin with two lemmas that iteratively apply Theorem 2 to partition the vertex set of a graph into monochromatic cliques. We continue with two lemmas that are convenient weakenings of a classical lemma of Kővári, T. Sós and Turán. Lastly, we end the section by stating a lemma involving a simple density calculation.

For the sake of simplicity, we denote $x^{A/\sqrt{\log x}} = e^{A\sqrt{\log x}}$ by $\tau(x)$, and $4^x\tau(x)/x = 4^x/x^{1-A/\sqrt{\log x}}$ by $T(x)$, where A is the constant appearing in Theorem 2.

Lemma 9. *Let p and n be positive integers such that $n \geq T(p)$. Then if K_n is 2-edge-colored, then we can partition $V(K_n)$ into classes $A_1, A_2, \dots, A_t, L$ such that the graph induced by A_i is monochromatic and $|A_i| = p$ for all i, and $|L| < T(p)$.*

Proof. We simply apply Theorem 2 to our graph, which gives a monochromatic complete subgraph of order p. Let the vertices of this subgraph be A_1. Then we apply Theorem 2 again to $G \setminus A_1$, to get A_2 and so on. We can repeat this process until we are left with less than $T(p)$ vertices, which will be our remainder set, L.

□

In the proof of Theorem 5, when applying Lemma 9 in the case $p = k$, we run into a problem when we cannot form vertex-disjoint monochromatic cliques of both colors in sufficient number (either because almost all of the monochromatic cliques are the same color, or because $n < T(k)$). However, this case is handled by our next lemma.

Lemma 10. *Let k, n, s be positive integers. Let K_n be 2-edge-colored. Then at least one of the next two statements hold:*

(i) *There exist disjoint subsets $R, B \subseteq V(K_n)$ such that R, respectively B, is union of vertex-sets of vertex-disjoint red, respectively blue, K_ks, and $|R|, |B| \geq s$.*
(ii) *There is a subset $X \subseteq V(K_n)$ with $|X| < 2s + k$ such that, in the subgraph induced by $V(K_n) \setminus X$, every monochromatic k-clique is of the same color.*

Proof. We apply our iterative process as in the proof of Lemma 9 as long as it is possible, but now we additionally require that the colors of the obtained monochromatic cliques alternate. That is, if there is a red k-clique in K_n then let its vertex-set be A_1; if there is a blue k-clique in the subgraph induced by $V(K_n) \setminus A_1$ then let its vertex-set be A_2; if there is a red k-clique in the subgraph induced by $V(K_n) \setminus (A_1 \cup A_2)$ then let its vertex-set be A_3; and so on. If we can continue it until $i = 2\lceil s/k \rceil$, then statement (i) obviously holds. Otherwise, for some $j \leq 2\lceil s/k \rceil - 1$, we cannot find an appropriate set A_{j+1}, hence every monochromatic k-clique is of the same color in the subgraph induced by $V(K_n) \setminus \cup_{\ell=1}^{j} A_\ell$. Let $X = \cup_{\ell=1}^{j} A_\ell$. Since

$$|X| = jk \leq \left(2\left\lceil \frac{s}{k} \right\rceil - 1\right) k < \left(2\frac{s+k}{k} - 1\right) k = 2s + k,$$

statement (ii) holds.

□

Our other useful tools will be two weaker but more conveniently applicable versions of a classical lemma in extremal graph theory, i.e., that of Kővári, T. Sós and Turán [7]. In their proof we shall use the notation $\{\alpha\}$ for the difference $\alpha - \lfloor \alpha \rfloor$, where α is arbitrary real number.

Lemma 11. *Let u, v be positive integers, and $G = (M, N)$ be a bipartite graph that contains no $K_{u,v}$. Let d be a positive integer with*

$$|N| > \frac{(v-1)\binom{|M|}{u}}{\binom{d}{u}}.$$

Then the next upper bound holds for the average degree in the class N:

$$\frac{|E(G)|}{|N|} \leq d - 1 + \frac{(v-1)\binom{|M|}{u}}{|N|\binom{d}{u}}.$$

Proof. Let $d_1, d_2, \ldots, d_{|N|}$ be the degree-sequence of vertices in N. In their classical paper, Kővári, T. Sós and Turán [7] used as a lemma that, if the conditions of our lemma hold, then, by the pigeon-hole principle, $\sum_{i=1}^{|N|} \binom{d_i}{u} \leq (v-1)\binom{|M|}{u}$.

Set $\bar{d} = \frac{|E(G)|}{|N|}$ and $t = \{\bar{d}\}|N|$. We use now, and also at the end of proof of next lemma, the trivial fact that every real number α is the following linear combination of its two integer neighbors:

$$\alpha = \{\alpha\}\left(\lfloor \alpha \rfloor + 1\right) + \left(1 - \{\alpha\}\right)\lfloor \alpha \rfloor.$$

Applying this for $\bar{d}$, we have

$$\begin{aligned}\sum_{i=1}^{|N|} d_i &= |E(G)| = |N|\bar{d} = |N|\left(\{\bar{d}\}\left(\lfloor \bar{d} \rfloor + 1\right) + \left(1 - \{\bar{d}\}\right)\lfloor \bar{d} \rfloor\right) \\ &= t\left(\lfloor \bar{d} \rfloor + 1\right) + \left(|N| - t\right)\lfloor \bar{d} \rfloor.\end{aligned}$$

Therefore, by the "convexity" of the binomial coefficients,

$$t\binom{\lfloor\bar{d}\rfloor+1}{u}+(|N|-t)\binom{\lfloor\bar{d}\rfloor}{u}\leq\sum_{i=1}^{|N|}\binom{d_i}{u},$$

and combining this with the Kővári-T. Sós-Turán lemma, we have (as observed originally by Guy [6])

$$t\binom{\lfloor\bar{d}\rfloor+1}{u}+(|N|-t)\binom{\lfloor\bar{d}\rfloor}{u}\leq(v-1)\binom{|M|}{u}. \tag{1}$$

It follows that $d\geq\lfloor\bar{d}\rfloor+1$, since a weakening of (1), and the conditions of our Lemma, give $|N|\binom{\lfloor\bar{d}\rfloor}{u}\leq(v-1)\binom{|M|}{u}<|N|\binom{d}{u}$. The only interesting case is $d=\lfloor\bar{d}\rfloor+1$, since if $d>\lfloor\bar{d}\rfloor+1$, then $\frac{|E(G)|}{N}=\bar{d}<\lfloor\bar{d}\rfloor+1\leq d-1$. For $d=\lfloor\bar{d}\rfloor+1$ the Lemma is true by another weakening of (1), i.e., $t\binom{\lfloor\bar{d}+1\rfloor}{u}\leq(v-1)\binom{|M|}{u}$, that yields

$$\frac{|E(G)|}{|N|}=\lfloor\bar{d}\rfloor+\frac{t}{|N|}\leq\lfloor\bar{d}\rfloor+\frac{(v-1)\binom{|M|}{u}}{|N|\binom{\lfloor\bar{d}\rfloor+1}{u}}=d-1+\frac{(v-1)\binom{|M|}{u}}{|N|\binom{d}{u}}.$$

□

We note that an earlier, even weaker and usually much more comfortable version of Guy's lemma, namely that the conditions above imply

$$|N|\binom{|E(G)|/|N|}{u}\leq(v-1)\binom{|M|}{u},$$

was shown first by Znám [12]. For our purpose it is not strong enough but Lemma 11 is easily derivable from it.

Now we shall show that we can substitute u into d and omit the condition on d in the previous lemma. In this way, we get a much weaker but simpler statement which will be easier to apply in the proof of Theorem 5.

Lemma 12. *Let u, v be positive integers, and $G=(M,N)$ be a bipartite graph that contains no $K_{u,v}$. Then the next upper bound holds for the average degree in the class N:*

$$\frac{|E(G)|}{|N|}\leq u-1+\frac{(v-1)\binom{|M|}{u}}{|N|}.$$

Proof. As earlier, let $\bar{d}=\frac{|E(G)|}{|N|}$. Set

$$z=\frac{(v-1)\binom{|M|}{u}}{|N|}.$$

Then, by (1),

$$\{\bar{d}\}\binom{\lfloor\bar{d}\rfloor+1}{u}+\left(1-\{\bar{d}\}\right)\binom{\lfloor\bar{d}\rfloor}{u}\leq z. \tag{2}$$

The left hand side of (2) is obviously strictly increasing in $\bar{d}$ for $\bar{d} \geq u-1$. Therefore if the bound to be proved does not hold, that is, $\bar{d} > u-1+z$, then

$$\begin{aligned} z &> \{u-1+z\}\binom{\lfloor u-1+z\rfloor+1}{u} + (1-\{u-1+z\})\binom{\lfloor u-1+z\rfloor}{u} \\ &= \{z\}\binom{u+\lfloor z\rfloor}{u} + (1-\{z\})\binom{u+\lfloor z\rfloor-1}{u} \\ &\geq \{z\}(\lfloor z\rfloor+1) + (1-\{z\})\lfloor z\rfloor \\ &= z, \end{aligned}$$

a contradiction.

□

Our final "lemma," which we state as a fact, involves a simple calculation of densities in a 2-edge-colored graph. As it is not hard to see, we shall omit the proof.

Fact 13. *Consider a K_t with $d_0\binom{t}{2}$ of the edges colored blue, and the others colored red. Let $E(K_t) = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, and set $\ell = \binom{t}{2}/|E_2|$. Also, for $i = 1, 2$, let $d_i|E_i|$ of the edges of E_i be blue, so that $d_0\binom{t}{2} = d_1|E_1| + d_2|E_2|$. Then $d_2 = d_0 + (\ell-1)(d_0-d_1)$.*

3. Proof of Theorem 5

We shall assume that $n > \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{m}}\right)T(m)$ and that there are no members of $\mathcal{S}_k$ in a 2-edge-colored K_n, and eventually reach a contradiction for sufficiently large C and k.

In the proof, without further reference, we shall frequently make use of two inequalities. The first is $\binom{k}{k/2} < 2^k/\sqrt{k}$, which is a straightforward consequence of Stirling's Formula for sufficiently large k. (In fact, it is also easy to see for every even k but we do not need this.) The second is $\tau(a) > \tau(b)$ for every $a > b > 1$, which holds since $\tau'(x) = Ae^{A\sqrt{\log x}}/2x\sqrt{\log x} > 0$ for every $x > 1$. Also, we shall refer to our colors as red and blue, and we will say that a clique is red (blue) if its all edges are red (blue).

Further, we can assume that $m \geq k-1$, since otherwise $d > 1/2$ and thus there are no 2-edge-coloring satisfying the conditions of the theorem. When $m \geq k-1$, we would like to now show that $n > \frac{4}{\sqrt{15}}T(k-1)$. For $m = k-1$, and $\varrho \geq 1/2$,

$$d \geq \frac{k-2}{2(k-1)} + \frac{k}{2(k-1)^2} = \frac{(k-2)(k-1)+k}{2(k-1)^2} > 1/2.$$

Therefore we can assume that $\varrho < 1/2$ and

$$n > \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{k-1}}\right)T(k-1) > \frac{4}{\sqrt{15}}T(k-1).$$

Finally, for $m > k-1$, $\varrho \leq 16/30$ yields

$$n > \left(1 + C/\sqrt{m}\right)T(m) > T(m) \geq T(k) > \frac{4}{\sqrt{15}}T(k-1).$$

We distinguish two cases according to whether our iterative search for monochromatic k-cliques yields a large number of both red and blue cliques or not. In the first case, we assume that there exist disjoint subsets $R_0, B_0 \subseteq V(K_n)$ such that R_0, respectively B_0, is union of vertex-sets of vertex-disjoint red, respectively blue, K_ks, and $|R_0|, |B_0| \geq 20T(k-1)/k^{3/2}$. Having defined these subsets of $V(K_n)$, we shall define another pair of subsets which are made up of monochromatic $(k-1)$-cliques. Our contradiction in this case will come by looking at densities of colors of edges between these various sets of vertices. By Lemma 9, there is a subset $A \subseteq V(K_n)$ that is union of vertex-sets of monochromatic, vertex-disjoint K_{k-1}s, and $n - |A| < T(k-1)$. We note that since, as we showed above, $n > \frac{4}{\sqrt{15}}T(k-1)$, $|A|/T(k-1) > n/T(k-1) - 1 > 4/\sqrt{15} - 1 > 1/50$.

Without loss of generality, we may assume that the majority of the monochromatic K_{k-1}s forming A are red. Thus there is a subset $R_1 \subseteq V(K_n)$ that is the union of vertex-sets of vertex-disjoint red K_{k-1}s, with $|R_1| \geq \frac{1}{2}|A| > \frac{T(k-1)}{100}$. Let d_r (and likewise d_b) be the density of red (likewise blue) edges between R_1 and B_0. For reasons that will become apparent later, we note that either

$$d_r \geq \frac{1}{2} - \frac{1}{k} + \frac{800}{k^{3/2}}$$

or

$$d_b \geq \frac{1}{2} + \frac{1}{k} - \frac{800}{k^{3/2}}.$$

We first consider the former subcase. Here we do not use our condition on $|B_0|$, rather only the fact that $|B_0| > 0$. As d_r is the density of red edges between R_1 and B_0, there exists a blue k-clique with red density between it and R_1 at least d_r. Let M be this blue clique, $N = R_1$ and G be the graph formed by the red edges between them. If G contains a $K_{k/2,\lfloor T(k/2)\rfloor}$, then we have a monochromatic $K_{k/2}$ in the class of size $\lfloor T(k/2)\rfloor$ by Theorem 2. The $K_{k/2,\lfloor T(k/2)\rfloor}$ restricted to this $K_{k/2}$ on the side of size $\lfloor T(k/2)\rfloor$ gives the desired member of $\mathcal{S}_k$. Otherwise, i.e., if G does not contain a $K_{k/2,\lfloor T(k/2)\rfloor}$, we apply Lemma 12. This leads to the following calculation:

$$\begin{aligned}
d_r k &\leq \frac{k}{2} - 1 + \frac{(\lfloor T(k/2)\rfloor - 1)\binom{k}{k/2}}{|R_1|} \\
&< \frac{k}{2} - 1 + \frac{4^{k/2}\tau(k/2)}{k/2} \cdot \frac{2^k}{\sqrt{k}} \cdot \frac{100}{T(k-1)} \\
&= \frac{k}{2} - 1 + \frac{2^{k+1}\tau(k/2)}{k} \cdot \frac{2^k}{\sqrt{k}} \cdot \frac{100(k-1)}{\tau(k-1)2^{2k-2}} \\
&< \frac{k}{2} - 1 + \frac{800}{\sqrt{k}},
\end{aligned}$$

for sufficiently large k. This contradicts our assumption that $d_r \geq \frac{1}{2} - \frac{1}{k} + \frac{800}{k^{3/2}}$.

In the other subcase, i.e., when $d_b \geq \frac{1}{2} + \frac{1}{k} - \frac{800}{k^{3/2}}$, the argument is very similar. We choose a red $(k-1)$-clique with blue density to R_1 at least d_b. Let M be this red $(k-1)$-clique, $N = B_0$ and G is the graph formed by the blue edges between them. As we did above, we are done if G contains a $K_{k/2,\lfloor T(k/2)\rfloor}$. Otherwise, apply Lemma 11 with $d = k/2 + 1$. Then

$$\begin{aligned}
d_b(k-1) &< \frac{k}{2} + \frac{(\lfloor T(k/2)\rfloor - 1)\binom{k-1}{k/2}}{|B_0|\binom{k/2+1}{k/2}} \\
&< \frac{k}{2} + \frac{4^{k/2}\tau(k/2)}{k/2} \cdot \frac{2^{k-1}}{\sqrt{k}} \cdot \frac{k^{5/2}}{20 \cdot 4^{k-1}\tau(k-1)} \cdot \frac{2}{k} \\
&< \frac{k}{2} + \frac{2}{5}.
\end{aligned}$$

Hence, for sufficiently large k,

$$\begin{aligned} d_b &< \frac{k/2+2/5}{k-1} \\ &= \frac{k/2+2/5}{k} \cdot \frac{k}{k-1} \\ &< \left(\frac{1}{2} + \frac{2/5}{k}\right)\left(1 + \frac{1}{k} + \frac{2}{k^2}\right) \\ &= \frac{1}{2} + \frac{9/10}{k} + \frac{7/5}{k^2} + \frac{4/5}{k^3} \\ &< \frac{1}{2} + \frac{1}{k} - \frac{800}{k^{3/2}}, \end{aligned}$$

giving the desired contradiction and completing the proof of the first case.

In the second case of the proof of the theorem, there are no subsets R_0, B_0 with R_0 (likewise B_0) the union of vertex-sets of vertex-disjoint red (likewise blue) K_ks, and $|R_0|, |B_0| \geq 20T(k-1)/k^{3/2}$. In this case, by applying Lemma 10 with $s = 20T(k-1)/k^{3/2}$, there is a set X with $|X| < 40T(k-1)/k^{3/2} + k$ such that, in the subgraph of K_n induced by $V(K_n) \setminus X$, every monochromatic k-clique is of the same color. Without loss of generality, we can assume that this color is red. We shall throw out X and calculate the density of blue edges in the remaining graph. We shall then iteratively consider monochromatic cliques of descending order, and if we are not able to find a blue k-clique in the remainder or a member of $\mathcal{S}_k$, then we reach a contradiction by forcing a blue density strictly greater than one.

We shall now need to define some notation for densities of colors among certain sets of edges. We let b be the density of blue edges in the original 2-edge-coloring of K_n, a be the density of blue edges among those incident to vertices in X, and b_0 be the density of blue edges in the graph formed by throwing out the vertices of X. Our immediate goal is to bound b_0 from below. By

Lemma 13 we can see that

$$\begin{aligned}
b_0 &= \frac{\binom{n}{2}}{\binom{n-|X|}{2}} \cdot b - \left(\frac{\binom{n}{2}}{\binom{n-|X|}{2}} - 1 \right) a \\
&= b + \left(\frac{\binom{n}{2}}{\binom{n-|X|}{2}} - 1 \right) (b - a) \\
&> b - \left(\frac{\binom{n}{2}}{\binom{n-|X|}{2}} - 1 \right) \\
&= b - \left(\frac{\binom{n}{2} - \binom{n-|X|}{2}}{\binom{n-|X|}{2}} \right) \\
&> b - \frac{2n|X|}{(n-|X|)(n-|X|-1)} \\
&> b - \frac{2n|X|}{(n-2|X|)^2} \\
&= b - \frac{2|X|}{n} \left(\frac{n}{n-2|X|} \right)^2 \\
&= b - \frac{2|X|}{n} \left(\frac{1}{1 - \frac{2|X|}{n}} \right)^2 \\
&\geq \frac{k-2}{2m} + \frac{\varrho k}{m^2} - \frac{2|X|}{n} \left(\frac{1}{1 - \frac{2|X|}{n}} \right)^2 . \qquad (3)
\end{aligned}$$

Thus, in order to get a useful lower bound on b_0, we need bound $|X|/n$ from above. To this end, we shall show that

$$\frac{|X|}{n} < \frac{40k}{m^{5/2}}. \qquad (4)$$

Indeed, for $m = k - 1$ and sufficiently large k,

$$\begin{aligned}
\frac{|X|}{n} &< \frac{40T(k-1)/k^{3/2} + k}{n} \\
&< \frac{10\sqrt{15}}{k^{3/2}} + \frac{k\sqrt{15}}{4T(k-1)} \\
&< \frac{40}{k^{3/2}} < \frac{40k}{m^{5/2}},
\end{aligned}$$

and for $m \geq k$ and sufficiently large k,

$$\begin{aligned}
\frac{|X|}{n} &< \frac{40T(k-1)/k^{3/2} + k}{n} \\
&< \frac{40T(k)}{nk^{3/2}} = \frac{40 \cdot 4^k \tau(k) k}{nk^{7/2}} \\
&\leq \frac{40 \cdot 4^m \tau(m) k}{nm^{7/2}} \\
&= \frac{40T(m)k}{nm^{5/2}} < \frac{40k}{m^{5/2}}.
\end{aligned}$$

Armed with our bound on $|X|/n$, we can see that for sufficiently large k, (4) implies

$$\left(\frac{1}{1-\frac{2|X|}{n}}\right)^2 < \left(\frac{1}{1-\frac{80k}{m^{5/2}}}\right)^2 < \frac{5}{4}. \tag{5}$$

Thus, we achieve our lower bound on b_0 by combining the inequalities derived in (3), (4), and (5) and see that

$$b_0 > \frac{k-2}{2m} + \frac{\varrho k}{m^2} - \frac{100k}{m^{5/2}}. \tag{6}$$

We shall now apply Lemma 9 iteratively, calculating blue densities at each step while assuming that there is no member of $\mathcal{S}_k$ in the graph. In this way, we arrive at a contradiction to the fact that the blue density must be at most one. Let $L_0 = V(K_n) \setminus X$. To begin, we apply Lemma 9 to $K_n[L_0]$ with $p = m$ to get monochromatic m-cliques and a leftover set L_1 with $|L_1| < T(m)$. We can assume that $|L_1| \geq T(m) - m$ since we can stop our search for m-cliques as soon as $|L_1| < T(m)$. Thus, in general, for $i \geq 0$, $T(m-i) - (m-i) \leq |L_{i+1}| < T(m-i)$ and $L_i \setminus L_{i+1}$ can be partitioned into vertex sets of monochromatic $(m-i)$-cliques.

We now establish a lower bound on $\frac{|L_i|}{|L_{i+1}|}$. Note that we can assume that m is sufficiently large, since we have shown that we may assume $m \geq k-1$ and that k is sufficiently large. Thus, the following lower bound holds for sufficiently large m and $1 \leq i \leq m/2$:

$$\begin{aligned}
\frac{|L_i|}{|L_{i+1}|} &> \left(\frac{4^{m-i+1}\tau(m-i+1)}{m-i+1} - (m-i+1)\right)\frac{m-i}{4^{m-i}\tau(m-i)} \\
&= 4\frac{(m-i)\tau(m-i+1)}{(m-i+1)\tau(m-i)} - \frac{(m-i+1)(m-i)}{4^{m-i}\tau(m-i)} \\
&> 4\left(\frac{m-i}{m-i+1}\right) - \frac{m^2}{4^{m/2}} \\
&= 4\left(1 - \frac{1}{m-i+1}\right) - \frac{m^2}{4^{m/2}} \\
&> 4\left(1 - \frac{2}{m}\right).
\end{aligned} \tag{7}$$

For $i \geq 0$, let b_i be the density of blue edges contained in L_i, and let a_i be the density of blue edges among edges entirely in L_i, but not entirely in L_{i+1}, or formally, the density of blue edges in $E(K_n[L_i]) \setminus E(K_n[L_{i+1}])$. As alluded to above, our proof will be completed by showing $b_i > 1$ for some i. We use Lemma 13 and thus need, in addition to (7), an upper bound on a_i. We shall show that if there is no member of $\mathcal{S}_k$ in the graph, then for $0 \leq i \leq 3\log_2 m$,

$$a_i < \frac{k-2}{2m} + \frac{ki}{2m^2} + \frac{2^{i+5}k}{m^{5/2}}. \tag{8}$$

To establish (8), first we note that if we consider a partition of $L_i \setminus L_{i+1}$ into vertex sets of monochromatic $(m-i)$-cliques, then the edges inside of $(m-i)$-cliques in $L_i \setminus L_{i+1}$ make up less than a $1/3^m$th proportion of the edges that contribute to a_i. This is because, for sufficiently large m,

$$|L_{i+1}| \geq T(m-i) - (m-i) = \frac{4^{m-i}\tau(m-i) - (m-i)^2}{m-i} > \frac{4^{m-i}}{m},$$

and using this, we see that

$$\begin{aligned}\frac{\frac{|L_i|-|L_{i+1}|}{m-i}\cdot\binom{m-i}{2}}{\binom{|L_i|}{2}-\binom{|L_{i+1}|}{2}} &= \frac{(|L_i|-|L_{i+1}|)\,(m-i-1)}{|L_i|\,(|L_i|-1)-|L_{i+1}|\,(|L_{i+1}|-1)}\\ &= \frac{m-i-1}{|L_i|+|L_{i+1}|-1}<\frac{m}{|L_{i+1}|}<\frac{m^2 4^i}{4^m}\\ &< \frac{m^2 4^{3\log_2 m}}{4^m}=\frac{m^8}{4^m}<\frac{1}{3^m}.\end{aligned}$$

To get the bound on a_i in (8), we shall assume the worst case, i.e., that all of the edges inside of these $(m-i)$-cliques are blue. Even in this case, it suffices to show that the density of the blue edges among the other edges used to calculate a_i must be less than

$$\frac{k-2}{2m}+\frac{ki}{2m^2}+\frac{2^{i+4}k}{m^{5/2}},$$

since then

$$a_i<\left(1-\frac{1}{3^k}\right)\left(\frac{k-2}{2m}+\frac{ki}{2m^2}+\frac{2^{i+4}k}{m^{5/2}}\right)+\frac{1}{3^k}<\frac{k-2}{2m}+\frac{ki}{2m^2}+\frac{2^{i+5}k}{m^{5/2}},$$

which is our desired bound.

Now we shall split the other edges that are used in calculating a_i (those not inside of $(m-i)$-cliques induced by the above partition in $L_i\setminus L_{i+1}$) into two bits: those between two $(m-i)$-cliques in $L_i\setminus L_{i+1}$ and those between $L_i\setminus L_{i+1}$ and L_{i+1}. Firstly, we consider edges of the latter type. In fact, we shall fix an arbitrary monochromatic $(m-i)$-clique in $L_i\setminus L_{i+1}$ and show the bound is true for edges between it and L_{i+1}. In this case, we can see that if we have a $K_{k/2,\lfloor T(k/2)\rfloor}$ with the $k/2$ part a subset of the $(m-i)$-clique, then we either have a member of $\mathcal{S}_k$, or we have a blue k-clique, a contradiction as we threw out the members of X and thus all blue k-cliques. So, we may assume that there is no blue $K_{k/2,\lfloor T(k/2)\rfloor}$. Now, we apply Lemma 12 to this set-up with M this monochromatic $(m-i)$-clique, $N=L_{i+1}$, and G the graph formed by the blue edges. The density we are considering in this case is

$$\frac{|E(G)|}{|M||N|}=\frac{1}{m-i}\cdot\frac{|E(G)|}{|N|}.$$

We note that, for $m-i>k$, $\binom{m-i}{k/2}<2\binom{m-i-1}{k/2}$, and for $m-i<k$, $\binom{m-i}{k/2}<\binom{m-i+1}{k/2}/2$; iterating these inequalities we have in both cases

$$\binom{m-i}{k/2}<2^{m-k-i}\binom{k}{k/2}<\frac{2^{m-i}}{\sqrt{k}}.$$

Therefore applying Lemma 12 for sufficiently large k and noting that $T(m-i)-(m-i) > T(m-i)/2$ gives

$$\begin{aligned}
\frac{|E(G)|}{|N|} &< \frac{k}{2} - 1 + \frac{(\lfloor T(k/2)\rfloor - 1)\binom{m-i}{k/2}}{|L_{i+1}|} \\
&< \frac{k}{2} - 1 + \frac{T(k/2)2^{m-i}}{\sqrt{k}(T(m-i)-(m-i))} \\
&< \frac{k}{2} - 1 + \frac{T(k/2)2^{m-i}}{\sqrt{k}(T(m-i)/2)} \\
&< \frac{k}{2} - 1 + \frac{2^k\tau(k/2)2^{m-i}(m-i)2}{(k/2)\sqrt{k}2^{2m-2i}\tau(m-i)} \\
&< \frac{k}{2} - 1 + \frac{2^{i+k-m+2}(m-i)}{k^{3/2}} \\
&< \frac{k}{2} - 1 + \frac{2^{i+3}k(m-i)}{m^{5/2}}.
\end{aligned}$$

Using the fact that, for $i < m/2$, $\frac{m}{m-i} = \frac{1}{1-i/m} < 1 + \frac{i}{m} + \frac{2i^2}{m^2}$, we have our desired upper bound for the density of blue edges between an arbitrary M monochromatic $(k-i)$-clique in $L_{i-1} \setminus L_i$ and $N = L_i$, since

$$\begin{aligned}
\frac{1}{m-i}\cdot\frac{|E(G)|}{|N|} &< \frac{1}{m-i}\left(\frac{k}{2} - 1 + \frac{2^{i+3}k(m-i)}{m^{5/2}}\right) \\
&= \frac{m}{m-i}\cdot\frac{k-2}{2m} + \frac{2^{i+3}k}{m^{5/2}} \\
&< \left(1 + \frac{i}{m} + \frac{2i^2}{m^2}\right)\frac{k-2}{2m} + \frac{2^{i+3}k}{m^{5/2}} \\
&< \frac{k-2}{2m} + \frac{ki}{2m^2} + \frac{2^{i+4}k}{m^{5/2}}.
\end{aligned}$$

Finally, in order to complete our bound on a_i, we would like to show that the same bound holds for edges between induced $(m-i)$-cliques in $L_i \setminus L_{i+1}$. This will work in a similar way to the above argument. We fix a $(m-i)$-clique in $L_i \setminus L_{i+1}$, say A. Since there once again cannot be a $K_{k/2,\lfloor T(k/2)\rfloor}$, we can apply Lemma 12 with $M = A$, $N = (L_{i-1} \setminus L_i) \setminus A$, and G the graph formed by blue edges. Then, the only difference from the above calculation is that now

$$|N| > |L_{i+1}|$$

and thus all of the remaining inequalities go through. Therefore, our desired bound on a_i, i.e., (8), holds.

Armed with our bound on a_i, we are ready to bound b_i. We would like to show that if there is no member of $\mathcal{S}_k$ in our 2-edge-colored K_n and C is sufficiently large, then for $1 \le i \le 3\log_2 m$,

$$b_i > \frac{2^i k}{m^{5/2}}. \tag{9}$$

For, then, if $i = 3\log_2 m$, this bound yields $b_i > k\sqrt{m} > 1$, our desired contradiction. In fact, we shall prove a stronger bound than (9) which is easier to establish by induction on i. We instead show

$$b_i > \frac{k-2}{2m} + \frac{ki}{2m^2} + \frac{k}{30m^2} + \frac{2^{i+6}k}{m^{5/2}}. \tag{10}$$

In order to base our induction, we must estimate $\frac{|L_0|}{|L_1|}$. Using first that $n > \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{m}}\right)T(m)$, then (4), we have

$$\begin{aligned}\frac{|L_0|}{|L_1|} &> \frac{n-|X|}{T(m)} \\ &> \left(1-\frac{|X|}{n}\right)\left(\frac{4}{\sqrt{30\varrho}}+\frac{C}{\sqrt{m}}\right) \\ &> \left(1-\frac{40k}{m^{5/2}}\right)\left(\frac{4}{\sqrt{30\varrho}}+\frac{C}{\sqrt{m}}\right) \\ &> \frac{4}{\sqrt{30\varrho}}+\frac{C}{2\sqrt{m}}, \end{aligned} \tag{11}$$

where the last inequality holds for sufficiently large k. By (6) and (8), we have

$$b_0 - a_0 > \frac{\varrho k}{m^2} - \frac{132k}{m^{5/2}} > \frac{\varrho k}{m^2} - \frac{200k}{m^{5/2}}. \tag{12}$$

For sufficiently large k the right hand side is positive, and hence

$$b_0 - a_0 > 0. \tag{13}$$

Combining Lemma 13, (13), (6), (11) and finally (12), for some C and sufficiently large k we have

$$\begin{aligned} b_1 &= \frac{\binom{|L_0|}{2}}{\binom{|L_1|}{2}}b_0 - \left(\frac{\binom{|L_0|}{2}}{\binom{|L_1|}{2}}-1\right)a_0 \\ &= b_0 + \left(\frac{\binom{|L_0|}{2}}{\binom{|L_1|}{2}}-1\right)(b_0-a_0) \\ &> b_0 + \left(\frac{|L_0|^2}{|L_1|^2}-1\right)(b_0-a_0) \\ &> \frac{k-2}{2m} + \frac{\varrho k}{m^2} - \frac{100k}{m^{5/2}} + \left(\left(\frac{4}{\sqrt{30\varrho}}+\frac{C}{2\sqrt{m}}\right)^2-1\right)\left(\frac{\varrho k}{m^2}-\frac{200k}{m^{5/2}}\right) \\ &> \frac{k-2}{2m} + \left(\frac{4}{\sqrt{30\varrho}}+\frac{C}{2\sqrt{m}}\right)^2\left(\frac{\varrho k}{m^2}-\frac{200k}{m^{5/2}}\right) \\ &> \frac{k-2}{2m} + \left(\frac{16}{30\varrho}+\frac{4C/\sqrt{30\varrho}}{\sqrt{m}}\right)\left(\frac{\varrho k}{m^2}-\frac{200k}{m^{5/2}}\right) \\ &= \frac{k-2}{2m} + \frac{k}{2m^2} + \frac{k}{30m^2} - \frac{320k}{3\varrho m^{5/2}} + \left(\frac{4C\sqrt{\varrho}}{\sqrt{30}} - \frac{800C/\sqrt{30\varrho}}{\sqrt{m}}\right)\frac{k}{m^{5/2}} \\ &> \frac{k-2}{2m} + \frac{k}{2m^2} + \frac{k}{30m^2-128m}\cdot\frac{30m^2-128m}{30m^2} + \frac{200k}{m^{5/2}} \\ &> \frac{k-2}{2m} + \frac{k}{2m^2} + \frac{k}{30m^2} + \frac{128k}{m^{5/2}}, \end{aligned}$$

thus (10) holds for $i = 1$.

Now let i be an integer with $2 \le i \le 3\log_2 k$ and suppose that (10) holds if we replace i by $(i-1)$. Combining this assumption and (8), we get

$$b_{i-1} - a_{i-1} > \frac{k}{30m^2} + \frac{2^{i+4}k}{m^{5/2}} \tag{14}$$

For sufficiently large k the right hand side is positive, and hence

$$b_{i-1} - a_{i-1} > 0. \tag{15}$$

Therefore, similarly to the beginning of our last calculation, we have

$$b_i > b_{i-1} + \left(\frac{|L_{i-1}|^2}{|L_i|^2} - 1\right)(b_{i-1} - a_{i-1}).$$

Combining (7), (14) and (10) for $i-1$, i.e., the inductive hypothesis, for sufficiently large k, we continue

$$\begin{aligned}
b_{i-1} \quad &+ \quad \left(\frac{|L_{i-1}|^2}{|L_i|^2} - 1\right)(b_{i-1} - a_{i-1}) \\
&> \quad b_{i-1} + \left(\left(4 - \frac{8}{m}\right)^2 - 1\right)(b_{i-1} - a_{i-1}) \\
&> \quad b_{i-1} + \left(15 - \frac{64}{m} + \frac{64}{m^2}\right)\left(\frac{k}{30m^2} + \frac{2^{i+4}k}{m^{5/2}}\right) \\
&> \quad \frac{k-2}{2m} + \frac{k(i-1)}{2m^2} + \frac{k}{30m^2} + \frac{2^{i+5}k}{m^{5/2}} \\
&\qquad + \left(15 - \frac{64}{m}\right)\left(\frac{k}{30m^2} + \frac{2^{i+4}k}{m^{5/2}}\right) \\
&= \quad \frac{k-2}{2m} + \frac{ki}{2m^2} + \frac{k}{30m^2} + \left(17 - \frac{64}{m}\right)\frac{2^{i+4}k}{m^{5/2}} \\
&> \quad \frac{k-2}{2m} + \frac{ki}{2m^2} + \frac{k}{30m^2} + \frac{2^{i+6}k}{m^{5/2}},
\end{aligned}$$

completing the proof of Theorem 5.

4. Proofs of the corollaries

In the proofs of the corollaries, we shall assume that C and k_0 are real numbers satisfying the statement of Theorem 5.

Proof of Corollary 6. First we assume that $k > k_0$. Let $m = m(k)$ be an integer such that

$$d(k) \geq \frac{k-2}{2m} + \frac{\varrho k}{m^2}.$$

for some $1/30 < \varrho \leq 16/30$. Then, by Theorem 5, the statement of the corollary holds for any integer $n(k)$ such that

$$n(k) > \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{m}}\right)T(m).$$

On the other hand, we must show the corollary is true when $k < k_0$. Let k_1 be the smallest even integer that is greater than k_0, and let $d_1 = \min\{d(k) | k < k_1, k \text{ is positive even integer}\}$. Again, let m be an integer such that

$$d_1 \geq \frac{k-2}{2m} + \frac{\varrho k}{m^2}.$$

for some $1/30 < \varrho \leq 16/30$. Then, by Theorem 5, if a 2-edge-coloring of K_n with density of each color at least d_1 does not yield any member of $\mathcal{S}_{k_1}$, then

$$n < \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{m}}\right)T(m). \tag{16}$$

If $k < k_1$, then $d(k) \geq d_1$, therefore it suffices for (16) that a 2-edge-coloring with densities at least $d(k)$ does not yield any member of $\mathcal{S}_k$.

□

Proof of Corollary 7. First we note that it suffices to prove the corollary for sufficiently large k, since this implies Corollary 7 for every k with perhaps a different choice of D.

Set $\mu = (k-2)/2d + 16/15 + 3/k$, $m = \lfloor \mu \rfloor$, and (using again the notation $\{\mu\}$ for $\mu - \lfloor \mu \rfloor$) $\varrho = 16/30 - \{\mu\}/2$. Note then that $m = (k-2)/2d + 2\varrho + 3/k$. First we shall show that, for sufficiently large k,

$$d \geq \frac{k-2}{2m} + \frac{\varrho k}{m^2}.$$

Indeed,

$$\begin{aligned} d - \frac{k-2}{2m} &= \frac{k-2}{2}\left(\frac{1}{(k-2)/2d} - \frac{1}{m}\right) \\ &= \frac{k-2}{2} \cdot \frac{2\varrho + 3/k}{m(k-2)/2d} \\ &> \frac{k}{2} \cdot \frac{2\varrho + 3/k}{m^2}\left(1 - \frac{2}{k}\right) \\ &= \frac{\varrho k}{m^2} + \frac{k}{2m^2}\left(\frac{3}{k} - \frac{4\varrho}{k} - \frac{6}{k^2}\right) \\ &> \frac{\varrho k}{m^2} \end{aligned}$$

for sufficiently large k.

Applying Theorem 5, we have that if there is a 2-edge-coloring satisfying the condition of the corollary, then, for sufficiently large k,

$$\begin{aligned} n &< \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{m}}\right) T(m) \\ &< \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{m}}\right) \frac{4^{m-k/2d}\tau(m)k}{\tau(k)m} \cdot \frac{4^{k/2d}\tau(k)}{k} \\ &< \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{(k-2)/2d}}\right) \frac{k4^{-1/d+2\varrho+3/k}}{(k-2)/2d} \cdot \frac{\tau(k/2d)}{\tau(k)} \cdot \frac{4^{k/2d}\tau(k)}{k} \\ &< \left(\frac{4}{\sqrt{30\varrho}} + \frac{C}{\sqrt{k/3d}}\right) 4^{3/k} 2d4^{-1/d+2\varrho} \cdot \frac{\tau(k/2d)}{\tau(k)} \cdot \frac{4^{k/2d}\tau(k)}{k} \\ &< \left(1 + \frac{C\sqrt{30\varrho}/4}{\sqrt{k/3d}}\right)\left(1 + \frac{6}{k}\right) e^{5/k} \frac{4}{\sqrt{30\varrho}} d4^{-1/d+1/2+2\varrho} \cdot \frac{\tau(k/2d)}{\tau(k)} \\ &\quad \cdot \frac{4^{k/2d}\tau(k)}{k} \\ &< \left(1 + \frac{C}{\sqrt{k}}\right) \frac{4^{2\varrho-1/15}}{\sqrt{30\varrho}} d4^{-1/d+47/30} \cdot \frac{\tau(k/2d)}{\tau(k)} \cdot \frac{4^{k/2d}\tau(k)}{k} \qquad (17) \end{aligned}$$

Now we show that, for $1/30 \leq \varrho < 16/30$,

$$4^{2\varrho-1/15}/\sqrt{30\varrho} \leq 1. \qquad (18)$$

Since equality holds for $\varrho = 1/30$ and $\varrho = 16/30$, we are done if the second derivative of the left hand side is positive for $1/30 \leq \varrho < 16/30$. Also, if we multiply the left hand side by a positive constant and show that this multiple has positive second derivative, we are also done. To this end, let $f(x) = 16^x x^{-1/2}$. Then, for $x > 0$, a simple calculation shows that $f''(x) > 0$.

For completing the proof, we need an upper bound bound on $\tau(k/2d)/\tau(k)$. For sufficiently large k,

$$\begin{aligned}
\frac{\tau(k/2d)}{\tau(k)} &= \frac{e^{A\sqrt{\log(k/2d)}}}{e^{A\sqrt{\log k}}} \\
&= e^{A\left(\sqrt{\log k+\log(1/2d)}-\sqrt{\log k}\right)} \\
&< e^{A\left(\sqrt{\log k}+\frac{\log(1/2d)}{2\sqrt{\log k}}-\sqrt{\log k}\right)} \\
&= e^{A\log(1/2d)/2\sqrt{\log k}} \\
&< 1+\frac{A\log(1/2d)}{\sqrt{\log k}}. \qquad (19)
\end{aligned}$$

Combining (17), (18), (19) and the fact that $d4^{-1/d+47/30} < 1$ for $d < 1/2$, we get the statement of the corollary for $D = A\log(1/2d)$ and sufficiently large k.

□

Proof of Corollary 8. First we note that it suffices to prove the corollary for sufficiently large k, since this, combined with either of the other two corollaries, implies Corollary 8 for every $k > 2c$, with a different choice of E.

We shall show that if the density of each color is at least $1/2 - c/k$, then each is at least $(k-2)/2m + \varrho k/m^2$ with $m = k+q-2$ and $\varrho = r - q^2/k$ for sufficiently large k. In other words, we shall show that

$$\frac{1}{2} - \frac{q/2-r}{k} \geq \frac{k-2}{2(k+q-2)} + \frac{(r-\varepsilon)k}{(k+q-2)^2}, \qquad (20)$$

if q is positive integer, $1/30 < r \leq 16/30$, $\varepsilon \geq q^2/k$ and k is sufficiently large. After doing this, we are able to apply Theorem 5. First we solve (20) for ε to get the following:

$$\varepsilon \quad \geq \quad \frac{(k+q-2)q(q-2)+2r(2k+q-2)(2-q)}{2k^2}. \qquad (21)$$

For sufficiently large k, the numerator of the right hand side is less than $2q^2k$, hence the right hand side is less than q^2/k. Therefore $\varepsilon \geq q^2/k$ suffices for (21), and thus (20).

Now we note that if $r > 1/30$ and k is sufficiently large, than $r - q^2/k > 1/30$. Thus, applying Theorem 5 for $m = k+q-2$ and $\varrho = r - q^2/k$, we have that if there is a 2-edge-coloring satisfying the condition of the corollary, then

$$n < \left(\frac{4}{\sqrt{30(r-q^2/k)}} + \frac{C}{\sqrt{k+q-2}}\right) T(k+q-2).$$

In order to complete the proof, we need to show that there is some E such that, for sufficiently large k, the last inequality implies the bound of the corollary on n. To this end, we shall show that

$$\frac{4^{q-1}}{\sqrt{30r}}\left(1+\frac{E}{\sqrt{k}}\right)T(k) > \left(\frac{4}{\sqrt{30(r-q^2/k)}}+\frac{C}{\sqrt{k+q-2}}\right)T(k+q-2). \tag{22}$$

In order to establish (22), we distinguish two cases depending on whether $q \geq 2$ or $q = 1$. For $q \geq 2$, we use the fact that $T(k+q-2) \leq 4^{q-2}T(k)$. This holds since, for sufficiently large k,

$$\begin{aligned} T(k+q-2)/T(k) &= 4^{q-2}\frac{\tau(k+q-2)k}{(k+q-2)\tau(k)} \\ &\leq 4^{q-2}\left(\frac{k+q-2}{k}\right)^{\frac{A}{\sqrt{\log k}}}\cdot\frac{k}{k+q-2} \\ &\leq 4^{q-2}. \end{aligned}$$

Hence for (22) for $q \geq 2$, it suffices that

$$\frac{4^{q-1}}{\sqrt{30r}}\left(1+\frac{E}{\sqrt{k}}\right)T(k) > \left(\frac{4}{\sqrt{30(r-q^2/k)}}+\frac{C}{\sqrt{k+q-2}}\right)4^{q-2}T(k),$$

that is,

$$\left(1+\frac{E}{\sqrt{k}}\right) > \sqrt{\frac{r}{r-q^2/k}}+\frac{C\sqrt{30r/16}}{\sqrt{k+q-2}}.$$

Rearranging, and denoting $r - q^2/k$ by ϱ again, we have

$$\frac{E}{\sqrt{k}}-\frac{C\sqrt{30r/16}}{\sqrt{k+q-2}} > \sqrt{\frac{\varrho+q^2/k}{\varrho}}-1.$$

Since

$$\varrho+\frac{q^2}{k} < \left(\sqrt{\varrho}+\frac{q^2}{2k\sqrt{\varrho}}\right)^2, \tag{23}$$

it suffices that

$$\frac{E}{\sqrt{k}}-\frac{C\sqrt{30r/16}}{\sqrt{k+q-2}} > \frac{\sqrt{\varrho}+q^2/2k\sqrt{\varrho}}{\sqrt{\varrho}}-1 = \frac{q^2}{2\varrho k}.$$

Since $r \leq 16/30$ and $q \geq 2$, for $E = 2C$ the left hand side is at least $C/\sqrt{k}$, hence the inequality holds for sufficiently large k.

For $q = 1$, (22) becomes

$$\frac{1}{\sqrt{30r}}\left(1+\frac{E}{\sqrt{k}}\right)T(k) > \left(\frac{4}{\sqrt{30(r-1/k)}}+\frac{C}{\sqrt{k-1}}\right)T(k-1).$$

Here we need $T(k) > 4(1-1/k)T(k-1)$. Indeed,

$$\frac{T(k)}{T(k-1)} = \frac{4(k-1)\tau(k)}{k\tau(k-1)} > 4\left(1-\frac{1}{k}\right).$$

Hence it suffices that

$$\frac{1}{\sqrt{30r}}\left(1+\frac{E}{\sqrt{k}}\right)4\left(1-\frac{1}{k}\right)T(k-1) > \left(\frac{4}{\sqrt{30(r-1/k)}}+\frac{C}{\sqrt{k-1}}\right)T(k-1),$$

that is,

$$\left(1+\frac{E}{\sqrt{k}}\right)\left(1-\frac{1}{k}\right) > \sqrt{\frac{r}{r-1/k}}+\frac{C\sqrt{30r/16}}{\sqrt{k-1}}.$$

Using (23) again, we have

$$\sqrt{\frac{r}{r-1/k}} = \sqrt{\frac{\varrho+1/k}{\varrho}} < 1+\frac{1}{2\varrho k}.$$

Thus it suffices that

$$\left(1+\frac{E}{\sqrt{k}}\right)\left(1-\frac{1}{k}\right) > 1+\frac{1}{2\varrho k}+\frac{C\sqrt{30r/16}}{\sqrt{k-1}},$$

which holds for $E = 2C$ and sufficiently large k.

□

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