Math cheatsheet for PHYS 191/192

Linear Algebra

Roots of quadratic equation $ax^2 + bx + c = 0$ are given by:
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Inverse operations:
if $y = x^n$, then $x = y^{1/n}$ (n-th root)
if $y = x^n$, then $\dfrac{1}{y} = x^{-n}$
if $x = \sin\theta$, then $\theta = \sin^{-1} x$ [similar expressions of cosine, tan, etc]
if $y = \ln x$, then $x = e^y$; if $y = \log x$, then $x = 10^y$

Trigonometry

Convert between radian and degree: $\theta \text{ (in radians)} = \theta \text{ (in degrees)} \times \dfrac{\pi}{180}$
Definition of angle using arc length and radius: $\theta = \dfrac{s}{r}$, where s is the arc length and r is the radius.
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; $\sec\theta = \dfrac{1}{\cos\theta}$
$\sin^2\theta + \cos^2\theta = 1$: This also means that $\sin\theta = \sqrt{1 - \cos^2\theta}$, and $\cos\theta = \sqrt{1 - \sin^2\theta}$
$\sec^2\theta - \tan^2\theta = 1$: This also means that $\sec\theta = \sqrt{1 + \tan^2\theta}$, and $\tan\theta = \sqrt{\sec^2\theta - 1}$

Trig functions of sums of angles:
$\sin(A + B) = \sin A \cos B + \cos A \sin B$, This also gives: $\sin 2A = 2 \sin A \cos A$
$\sin(A - B) = \sin A \cos B - \cos A \sin B$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$, This also gives: $\cos 2A = \cos^2 A - \sin^2 A$
$\cos(A - B) = \cos A \cos B + \sin A \sin B$

Using the $\cos 2A$ formula and $\sin^2\theta + \cos^2\theta = 1$ above we can also write:
$\cos^2 A = \dfrac{1 + \cos 2A}{2}$; and $\sin^2 A = \dfrac{1 - \cos 2A}{2}$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$; $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Calculus

Limit: $\sin\theta \approx \theta, \cos\theta \approx 1, \text{ and } \tan\theta \approx \theta$ for $\theta \to 0$

DIFFERENTIATION:

$\frac{d}{dx}C = 0$, where C is constant.

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}Cx^n = C\frac{d}{dx}x^n = nCx^{n-1}$$

Most generally: $\frac{d}{dx}(C_1x^n + C_2x^m + \ldots) = nC_1x^{n-1} + mC_2x^{m-1} + \ldots$

Basically, differentiation operation distributes into each term in a sum, and only acts on the function w.r.t which the differentiation is done (in this case x). **Same goes for integration**.

$$\frac{d}{dx}\sin x = \cos x; \frac{d}{dx}\cos x = -\sin x; \frac{d}{dx}\tan x = \sec^2 x; \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\sin^n x = \frac{d}{d(\sin x)}\sin^n x \,.\, \frac{d(\sin x)}{dx} = n\sin^{n-1}x \,.\, \cos x$$

similarly: $\frac{d}{dx}\cos^n x = \frac{d}{d(\cos x)}\cos^n x \,.\, \frac{d(\cos x)}{dx} = -n\cos^{n-1}x \,.\, \sin x$

and so forth for the other trig functions. This technique is also called the chain rule. In its most general form it is written as:

$\frac{d}{dx}f(u) = \frac{d}{du}f(u) \,.\, \frac{du}{dx}$. You can convince yourself by choosing $u = \sin x$, and $u = \cos x$ to derive the expressions of differentiation of $\sin^n x$ and $\cos^n x$ above using the chain rule.

Inverse trig functions: $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$; $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}e^{\alpha x} = \alpha e^{\alpha x}$; $\frac{d}{dx}\ln x = \frac{1}{x}$

Double derivative: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Meaning of derivative:

If an equation of a curve is given as $y = f(x)$, then the derivative $\frac{dy}{dx}$ is the slope of the curve.

For the simplest case the equation of line $y = mx + c$, the slope is $\frac{d}{dx}(mx + c) = m$, which we know is really the slope of the line.

Maxima and minima: To find the maxima or minima of a function $f(x)$, we first set its derivative to be zero, $\frac{df(x)}{dx} = 0$. We solve for the x. We then calculate the double derivative $\frac{d^2f}{dx^2}$ at that value of x. If the answer is positive then said point is a minima, and its a maxima if the double derivative is negative.

Product rule: $\frac{d}{dx}(f(x)\,.\,g(x)) = \frac{df}{dx}g + f\frac{dg}{dx}$

Quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{(g(x))^2}$

INTEGRATION

The integral of the function is the inverse operation of differentiation. Or in other words if you differentiate a function $f(x)$ to get another function $F(x)$ then the integral of that function will give you back the original function $f(x)$ (with a constant term).

STANDARD FORMS:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$\int \sin x dx = -\cos x + C$; $\int \cos x dx = \cos x + C$; $\int \sec^2 x dx = \tan x + C$;

$\int \sec x \tan x dx = \sec x + C$

$\int \sin nx dx = -\frac{\cos nx}{n} + C$, and likewise for the other trig functions.

$$\int e^{\alpha x} dx = \frac{e^x}{\alpha} + C$$

$$\int \frac{1}{x} dx = \ln x + C$$

INTEGRATION BY PARTS:

$$\int u(x)\ v(x)\ dx = v(x) \int u(x)\ dx - \int dx \left(\frac{dv}{dx} \int u(x)\ dx \right)$$

choose $u(x)$ as the part of the function that is easy to integrate, and $v(x)$ as the part of the function that becomes simpler upon differentiating.

Example: $\int x \sin x\ \ dx$

choose $u(x) = \sin x$, and $v(x) = x$. This is because we know how to integrate $\sin x$ and differentiating x gives 1 making it simpler.

METHOD OF SUBSTITUTION:

If an integral is not given in a standard form, we can try to reduce it to a standard form by clever substitutions. When substituting, all variables must be substituted, even the dx terms.

Example: $\int \frac{1}{\sqrt{1 - x^2}} dx$

Let us substitute $x = \sin \theta$ then $\sqrt{1 - x^2} = \cos \theta$, and $dx = d(\sin \theta) = \cos \theta\ \ d\theta$. Thus the integral becomes

$$\int \frac{1}{\sqrt{1 - x^2}} dx \rightarrow \int \frac{\cos \theta\ d\theta}{\cos \theta} = \int d\theta = \theta + C$$

But since $\theta = \sin^{-1} x$, therefore: $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + C$

We can see that must be true because in the differentiation section we saw that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$.

DEFINITE INTEGRAL:

if $F(x) = \int f(x)\ dx$ then,

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Geometrically, this gives the area under the curve of the function $F(x)$ between $x = a$ and $x = b$