Econ674

Economics of Natural Resources and the Environment

Session 6

Dynamic Optimization in Natural Resources

Basic Issues in Dynamic Optimization

Lagrangian formulations can be used to solve both static and dynamic optimization problems. Such problems can be formulated either in discrete or continuous time.

We begin with discrete optimization to elaborate on optimal control theory and the maximum principle, after which we restate the problem in a continuous time format.

Let $t = 0,1,..., T$ *be the* set of time periods for the dynamic allocation problem, where $t = 0$ is the present and $t = T$ is the terminal (last) period. We then incorporate the method of Lagrangian multipliers:

x_t represents a state variable (describing the system in period t)
y_t *a* control/instrument variable in period t,
$V = V(x_t, y_t, t)$ net economic return/objective function in period t
$F(x_T)$, a function showing the value of alternative levels of the x_t at T
$x_{t+1} - x_t = f(x_t, y_t)$ a difference equation defining the change in the state variable from t to $(t + 1)$, $t = 0,...,T-1$

Time has been partitioned into a finite number of discrete periods (T+1) (but an infinite horizon can be allowed by letting $T \to \infty$)

We restrict ourselves to a single state, single control variable case for simplicity; but the problem may be readily generalized to I state variables an J control variables

The objective function V(.) may have the period index t as a variable but the difference equation does not, thus f(.) is said to be autonomous.

Use of Lagrangian Multipliers

$$\max_{\{y_t\}} \sum_{t=0}^{T-1} V(x_t, y_t, t) + F(x_T) \equiv V(x_0, y_0, 0) + V(x_1, y_1, 1) + \ldots. +$$
$$V(x_{T-1}, y_{T-1}, T-1) + F(x_T)$$
$$\text{subject to } x_{t+1} - x_t = f(x_t, y_t)$$
$$x_0 = a \text{ given} \qquad \textbf{Eq (1)}$$

The problem is to maximize the sum of intermediate values and the value associated with terminal state x_T, subject to the difference equation describing changes in x_t over the horizon, assuming $x_0 = a$ (fixed)

The problem then becomes determining the optimal values for *yt*, *t* = 0, 1, ..., *T*-1 which will, via the difference equation, imply values for *xt*, *t* = 1, …, *T*-1.

The solution of such a problem is a path-determined as a function of time, or in our discrete-time problem, in tabular form.

The Lagrangian for the discrete-time problem can be stated as:

$$L = V(x_0, y_0, 0) + \lambda_1\left(x_0 + f(x_0, y_0) - x_1\right) + V(x_1, y_1, 1) + \lambda_2\left(x_1 + f(x_1, y_1) - x_2\right) \ldots +$$
$$V(x_{T-1}, y_{T-1}, T-1) + \lambda_T\left(x_{T-1} + f(x_{T-1}, y_{T-1}) - x_T\right) + F(x_T)$$

or

$$L = \sum_{t=0}^{T-1} \left\{V(\cdot) + \lambda_{t+1}\left(x_t + f(x_t, y_t) - x_{t+1}\right)\right\} + F(x_T) \qquad \textbf{Eq.} \quad (\textbf{2})$$

Use of Lagrangian Multipliers - 1

The problem has T constraint equations for the t = 0, ..., T-1 periods

λ_{t+1} is a multiplier associated with X_{t+1}

Where lambda t+1 is a multiplier associated with x t+1. Because there are T such constraint equations (t=0,…, T-1) it is appropriate to include them within the summation operation,

With no non-negativity constraints, the first-order conditions (FOC's) require:

$$\frac{\partial L}{\partial y_t} = \frac{\partial V}{\partial y_t} + \lambda_{t+1}\frac{\partial f}{\partial y_t} = 0 \quad t = 0,1,\ldots,T-1 \quad \mathbf{Eq(3)}$$

$$\frac{\partial L}{\partial x_t} = \frac{\partial V}{\partial x_t} + \lambda_{t+1}\left(1 + \frac{\partial f}{\partial x_t}\right) - \lambda_t = 0 \quad t = 1,\ldots,T-1\,\mathbf{Eq\ (4)}$$

$$\frac{\partial L}{\partial x_T} = -\lambda_T + \frac{\partial F}{\partial x_T} = 0 \quad \mathbf{Eq(5)}$$

$$\frac{\partial L}{\partial \lambda_{t+1}} = x_t + f(\cdot) - x_{t+1} = 0 \quad t = 0,,\ldots,T-1\,\mathbf{Eq.(6)}$$

All partials are straightforward except equation (4). In taking the partial of L with respect to xt on looks at where x_t appears in the t th term of the summation. This accounts for the first two expressions on the RHS of (4). If, however, one were to back up to the (t-1) term one would also find a-xt pre-multiplied by lambda t, hence the third expression –lambda t in (4).

Use of Lagrangian Multipliers - 2

To facilitate and compare the discrete solution with continuous time problems, we re-write the first-order conditions as follows:

$$\frac{\partial V}{\partial y_t} + \lambda_{t+1}\frac{\partial f}{\partial y_t} = 0 \quad t = 0,1,\ldots,T-1 \ \mathbf{Eq.(7)}$$

$$\lambda_{t+1} - \lambda_t = -\left(\frac{\partial V}{\partial x_t} + \lambda_{t+1}\frac{\partial f}{\partial x_t}\right) \quad t = 1,\ldots,T-1 \ \mathbf{Eq.(8)}$$

$$x_{t+1} - x_t = f(\cdot) \qquad \text{Eq.}(9)$$

$$\lambda_T = \frac{\partial F}{\partial x_T} \quad \mathbf{Eq.(10)}$$

$$x_0 = a \ \mathbf{Eq.(11)}$$

Equations (7) will typically define a marginal condition that y_t must satisfy. In the dynamic allocation problem the result comes out with two terms:

$\partial V(\cdot)/\partial y_t$ has the interpretation of a net marginal benefit *in period t.*

$\lambda_{t+1}(\partial f / \partial y_t)$ reflects the influence of y_t on the change in the state variable; since an increase in y_t reduces x_{t+1}, this reflects the *user cost.*

At the optimal solution of the problem λ^*_{t+1} reflects effects of increases in X_{t+1} on the *remaining time* (t+1,...,T). A 2nd cost must be considered in undertaking an incremental action today- the marginal losses that might be incurred over the remaining future.

Note: In equation 7 there is a term not found in static problems which is the one involving lambda

Use of Lagrangian Multipliers - 3

•The difference equation in 8 must hold through time and relate changes in the Lagrange multiplier with terms of partials of xt. They show how the multiplier changes optimally over t

•Equation 9 is a restatement of the difference equation for the state variable.

•Equations 10 and 11 are the boundary conditions and define the terminal value of the multiplier and the initial condition on the state variable

•Eq. 7-11 is a system of (3T + 1) equations in (3T + 1) unknowns: yt for t=0, ...,T-1; xt for t=0,...,T; and 8t for t = 1,..., T .

•It may be possible to solve the system simultaneously for yt, xt and 8t.

•But it may be more efficient to solve the system by algorithm.

•If xt, yt, and 8t are restricted to being nonnegative one must formulate the appropriate Kuhn-Tucker conditions with possible solution via nonlinear programming.

[Note: If 8t could also be specified initially, the system 9-11 could be completely solved by numerical iteration starting at t = 0. The expression in equation 8 can be given a nice, intuitive interpretation within the context of harvesting a renewable resource and we postpone its discussion till then.

For equations 10 and 11, the boundary conditions are referred to as a ‘split’ since one condition is an initial condition and the other is a terminal condition]

Fixed Versus Free Terminal Time and Terminal State

•"*Fixed-time, free-state*": Specify the horizon but not the x_T: Eq 1

•"*Free-time*": Decision-maker determines the optimal horizon (T*).

•Differential condition may be used to determine optimal T, T*, in continuous-time.

•No differential relationship in discrete-time; the decision-maker explores different horizons, determine the optimal behavior for each horizon, and compare the sum of net economic returns.

•"*Restricted free-time*": The problem imposes a constraint on the length of horizon (e.g., t' ≤T* ≤ t'' where t' and t''are given).

•In relation to free time, it may be noted that in continuous-time the optimal horizon might be determined by a differential condition (partial of L wrt T equals 0). In discrete-time there would be no differential relationship and the decision maker would have to explore horizons of different length, determine the optimal behaviour for each horizon (T), and then compare the sum of net economic returns.

•As an example of a stationary state with finite horizon: it may be optimal for the manager of a mine to deplete his reserves before the end of a given planning horizon.

"Infinite horizon" : $T \to \infty$ is allowed . Is the solution a steady (stationary) state or not? If a steady state is possible from period J onwards then

yt = y*, xt = x*, and 8t = 8* for all t $\in \vartheta$

Eq (12)

The solution to finite/fixed horizon problems may also lead to a stationary state.

"Terminal surface" models: The decision-maker has some freedom in the selection of T and xT;

An example where a steady state is possible with finite horizon problems is when a mine manager finds it optimal to deplete his reserves before the end of a given planning horizon

The Infinite Horizon Problem and the Steady State

Consider the following problem:

$$\max_{\{y_t\}} \sum_{t=0}^{\infty} V(x_t, y_t)$$

$$\text{subject to } x_{t+1} - x_t = f(x_t, y_t)$$

$$x_0 = a \text{ given} \qquad \textbf{Eq.(13)}$$

t not an argument of *V* (.) and since $T \to \infty$ there is no final function. Under these conditions, the Lagrangian becomes:

$$L = \sum_{t=0}^{\infty} \{V(\cdot) + \lambda_{t+1}(x_t + f(x_t, y_t) - x_{t+1})\} \quad \textbf{Eq.(14)}$$

The first order conditions are:

$$\frac{\partial V}{\partial y_t} + \lambda_{t+1} \frac{\partial f}{\partial y_t} = 0 \qquad t = 0,1,\ldots \quad \textbf{Eq.(15)}$$

$$\lambda_{t+1} - \lambda_t = -\left(\frac{\partial V}{\partial x_t} + \lambda_{t+1} \frac{\partial f}{\partial x_t}\right) \quad t = 0,1,\ldots \quad \textbf{Eq}(16)$$

$$x_{t+1} - x_t = f(\cdot) \qquad t = 0,1,\ldots \quad \textbf{Eq.(17)}$$

In a steady state, with constant yt, xb, and 8t, Equations 15-17 become a 3-equation system

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0 \quad \textbf{Eq.}(18)$$

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial f}{\partial x} \quad \textbf{Eq}(19)$$

$$f(\cdot) = 0 \quad \textbf{Eq.}(20)$$

The Infinite Horizon Problem and the Steady State - 1

- •The above can be solved for the steady-state optimum (y*, x*, 8*).
- •If a steady-state optimum exists for an infinite horizon problem, if it is unique, can be solved and the system is not currently in a steady-state optimum (i.e., x0 … x*), then what is the best way to get there?
- •Assuming x* is reachable from x0, there are two types of optimal approach paths from x0 to x*.
- •First: the asymptotic approach, i.e., xt6 x* as t 6%.
- •Second: the most rapid approach path (MRAP), xt is driven to x* as rapidly as possible, which will often involve a "bang-bang" control where yt during the MRAP assumes some maximum or minimum value.

[Note: by eliminating lambda from equations 18 and 19 an solving 20 for y as a function of x it is often possible to obtain a single equation in the variable x*.]

Conditions for the Most Rapid Approach Path to be Optimal:

(a) via constraint-substitution: $V(x_t, y_t)$ must be expressed as additively separable function in x_t and x_{t+1} and
(b) via proper indexing: make the problem equivalent to optimization of $\sum_{t=1}^{\infty} w(x_t)$ with $w(.)$ quasi-concave.

There are many intuitive specifications for dynamic problems which satisfy the necessary and sufficiency conditions for MRAP to be optimal.

Specification of the Hamiltonian Function:

Define the Hamiltonian as:

$$H\left(x_t, y_t, \lambda_{t+1}, t\right) = V(x_t, y_t, t) + \lambda_{t+1}\left(f(x_t, y_t)\right) \quad \mathbf{Eq}(\mathbf{2}1)$$

This allows us to write the FOC given earlier directly as partials of the Hamiltonian. First, note that the Lagrangian expression (2) may be written in terms of the Hamiltonian as follows:

$$L = \sum_{t=0}^{T-1} \left\{H + \lambda_{t+1}\left(x_t - x_{t+1}\right)\right\} + F\left(x_T\right) \qquad 22$$

$$= \sum_{t=0}^{T-1} \left\{V(x_t, y_t, t) + \lambda_{t+1}\left(f(x_t, y_t)\right) + \lambda_{t+1}\left(x_t - x_{t+1}\right)\right\} + F\left(x_T\right)$$

$$= \sum_{t=0}^{T-1} \left\{V(x_t, y_t, t) + \lambda_{t+1}\left(x_t + f(x_t, y_t) - x_{t+1}\right)\right\} + F\left(x_T\right)$$

The corresponding first-order conditions are:

$$\frac{\partial L}{\partial y_t} = \frac{\partial H(\cdot)}{\partial y_t} = 0 \qquad t = 0,\ldots,T-1 \quad 23$$

$$\frac{\partial L}{\partial x_t} = \frac{\partial H(\cdot)}{\partial x_t} + \lambda_{t+1} - \lambda_t = 0 \qquad t = 1,\ldots,T-1 \quad 24$$

$$\frac{\partial L}{\partial x_T} = -\lambda_T + F'(\cdot) = 0 \qquad 25$$

$$\frac{\partial L}{\partial \lambda_{t+1}} = \frac{\partial H(\cdot)}{\partial \lambda_{t+1}} + x_t - x_{t+1} = 0 \qquad t = 0,\ldots,T-1 \quad 26$$

The most familiar form of these conditions is written as the set:

$$\frac{\partial H(\cdot)}{\partial y_t} = 0; \quad \lambda_{t+1} - \lambda_t = -\frac{\partial H(\cdot)}{\partial x_t}; \quad x_{t+1} - x_t = \frac{\partial H(\cdot)}{\partial \lambda_{t+1}} \qquad \mathbf{Eq}(\mathbf{27})$$

$$\lambda_T = F'(\cdot) = 0 \qquad x_0 = a$$

An Example with a Hamiltonian Formulation

A manager of a mine wishes to determine the optimal production schedule for t = 0,... ,9: mine will be shut down/abandoned at t = 10.

p = price of a unit of ore=1, and

the cost of extracting y_t is $c_t = y_t^2 / x_t$

x_t is *remaining reserves* at the beginning of period *t*.

Net revenue is $\pi_t = py_t - y_t^2 / x_t = y_t\left[1 - y_t / x_t\right]$

the difference equation describing the change in remaining reserves is: $x_{t+1} - x_t = -y_t$

Initial reserves are assumed given with x_0 = 1,000.

Note: the original problem stated in eq. 1 is an example of a subclass of control problems called open-loop problems. The solution of such a problem is a control trajectory y*t determined as a function of time, or in our discrete-time problem, in tabular form. Knowing y*t and x0 one can use the difference equation x t+1= xt +f(.) to solve forward for the optimal trajectory xt, denoted x*t.

Note: in this problem there is no final function an any units of x remaining in period 10 must be worthless. Note also that this is a fixed-time free-state problem and that the first order conditions represent a system of 31 equations in 31 unknowns: yt for t=0,1,…, 9, xt for t=0,1,…,10, and lambda t for t=1,2,…, 10. Solution of this problem is most easily accomplished by defining zt=yt/xt.

Maximization of the sum of net revenues subject to reserve dynamics leads to the Hamiltonian:

$$H(\cdot) = y_t\left[1 - y_t / x_t\right] - \lambda_{t+1} y_t$$

$$\text{FOC given by} \rightarrow$$

An Example with a Hamiltonian Formulation - 1

$$\frac{\partial H(\cdot)}{\partial y_t} = 1 - 2y_t / x_t - \lambda_{t+1} = 0 \quad t = 0,\ldots,9$$

$$\lambda_{t+1} - \lambda_t = -\frac{\partial H(\cdot)}{\partial x_t} = -\frac{y_t^2}{x_t^2} \quad t = 1,\ldots,9$$

$$x_{t+1} - x_t = -y_t \quad t = 0,\ldots,9$$

$$x_0 = 1000, \quad \lambda_{10} = F'(\cdot) = 0$$

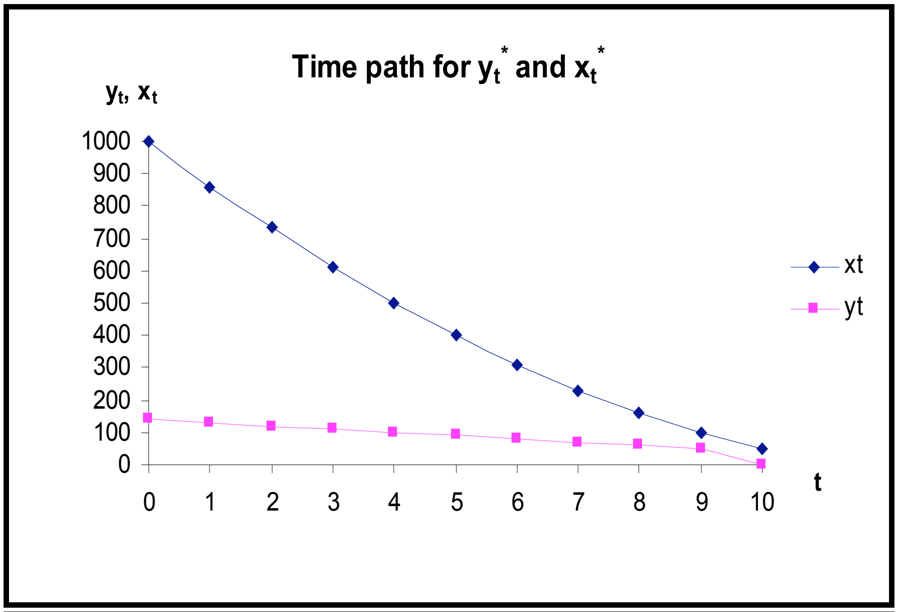
In this problem there is no final function and any units of x remaining in period 10 must be worthless. Note also that this is a fixed-time free-state problem.

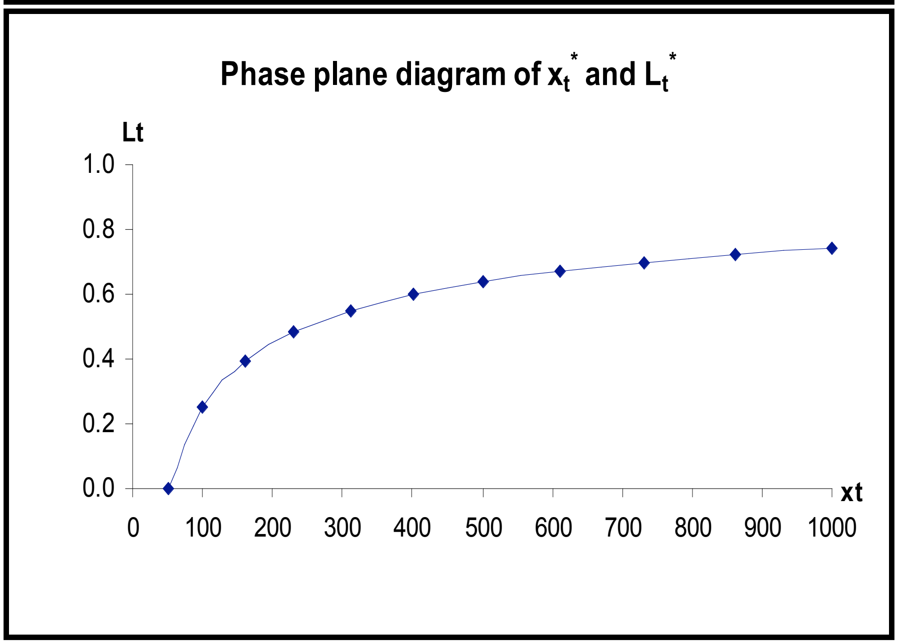
The FOC represent a system of 31 equations in 31 unknowns: yt for t = 0,1,. . . ,9, xt for t = 0,1,. . . , 10, and 8t for t = 1,2,...,10.

For a solution see the table, the time path of the control and state variables, and the phase diagram

t	$_t$	y_t	x_t
0.00	0.74	138.90	1000.00
1.00	0.72	129.32	861.10
2.00	0.70	119.69	731.78
3.00	0.67	109.99	612.09
4.00	0.64	100.23	502.10
5.00	0.60	90.39	401.87
6.00	0.55	80.45	311.48
7.00	0.48	70.39	231.04
8.00	0.39	60.24	160.64
9.00	0.25	50.20	100.40
10.00	0.00	0.00	50.20

An Example with a Hamiltonian Formulation - 2





Solution of this problem is most easily accomplished by defining zt=yt/xt. Evaluating dH/dyt at t=9 implies z9=0.5 (since lambda 10=0). Evaluating the expression for lambda t+1 – lambda t at t=9 implies lambda 9=(z9) squared=0.25. Knowing lambda 9 we can return to dH/dyt to solve for z8, then back down to the second equation for lambda 8, and so forth. The last step in the recursion gives us z0=0.1389 and lambda 0=0.7415. Knowing that x0=1000 we can solve for y0=x0z0=138.90 and x1=x0-y0=861.10. Knowing x1 we can solve for y1=x1z1=129.32, x2=x1-y1=731.78, and so forth. Conrad and Clark present results using Basic but the same results could be found using Excel as in Conrad (1999).

Continuous Time and the Maximum Principle

When t is allowed to be continuous the optimization interval becomes 0 # t # T

The difference equation describing the change in the state variable is replaced by the differential equation

dx(t)/dt = $\dot{x}$ = f(A).

The continuous-time analogue to the discrete problem is

$$\max \int_0^T V(x(t), y(t), t)\,dt + F\big(x(T)\big)$$
$$\text{subject to } \dot{x} = f(x(t), y(t))$$
$$x(0) = a \text{ given} \qquad \mathbf{EQ.(1)}$$

The integration replaces the discrete-time summation

By convention, the continuous-time variables parenthesize *t* as opposed to subscripting.

In the continuous-time problem it is necessary to assume *x*(*t*) is continuous and *y*(*t*) piecewise continuous.

We can form a Lagrangian expression for this problem as well

$$L = \int_0^T \left[V(\cdot) + \lambda(t)\big(f(\cdot) - \dot{x}\big)\right]dt + F\big(x(T)\big)$$
$$= \int_0^T \left[V(\cdot) + \lambda(t) f(\cdot) - \lambda(t)\dot{x}\right]dt + F\big(x(T)\big) \quad \mathbf{EQ(2)}$$

Integrate $-\lambda(t)\dot{x}$ by parts to obtain:

$$= \int_0^T \dot{\lambda} x(t)\,dt - \big(\lambda(T)x(T) - \lambda(0)x(0)\big)$$

Continuous Time and the Maximum Principle - 1

which upon substitution into (2) gives:

$$L = \int_0^T \left[V(\cdot) + \lambda(t) f(\cdot) + \dot{\lambda} x(t) \right] dt + F(\cdot) - \left(\lambda(T) x(T) - \lambda(0) x(0) \right) \quad \mathbf{EQ(3)}$$

Now define the continuous-time Hamiltonian as:

$$H\left(x(t), y(t), \lambda(t), t\right) = V(x(t), y(t), t) + \lambda(t)\left(f(x(t), y(t))\right) \quad \mathbf{EQ(4)}$$

Next, re-arrange the Lagrangian expression as:

$$L = \int_0^T \left[H(\cdot) + \dot{\lambda} x(t) \right] dt + F(\cdot) - \left(\lambda(T) x(T) - \lambda(0) x(0) \right) \quad \mathbf{EQ(5)}$$

The first-order conditions may be derived in the following heuristic manner. Consider a change in the control trajectory from *y*(*t*) to $y(t) + \Delta y(t)$ which causes a change in the state trajectory from *x*(*t*) to $x(t) + \Delta x(t)$

The change in the Lagrangian is:

$$\Delta L = \int_0^T \left[\frac{\partial H(\cdot)}{\partial y(t)} \Delta y(t) + \frac{\partial H(\cdot)}{\partial x(t)} \Delta x(t) + \dot{\lambda} \Delta x(t) \right] dt + \left[F'(\cdot) - \lambda(T) \right] \Delta x(T) \quad \mathbf{EQ}(6)$$

For a max the change in the Lagrangian must vanish for any $\{\Delta y(t)\}$

Thus: $$\frac{\partial H(\cdot)}{\partial y(t)} = 0; \quad \dot{\lambda} = -\frac{\partial H(\cdot)}{\partial x(t)}; \quad \lambda(T) = F'(\cdot) \quad \mathbf{EQ(7)}$$

From the definition of H(.), we may write:

$$\frac{\partial H}{\partial \lambda_t} = \left(f(x(t), y(t)) \right) = \dot{x}$$

Continuous Time and the Maximum Principle - 2

and taking into account the initial condition, we may write the necessary conditions in their entirety as:

$$\frac{\partial H(\cdot)}{\partial y(t)} = 0; \quad \dot{\lambda} = -\frac{\partial H(\cdot)}{\partial x(t)}; \quad \dot{x} = \frac{\partial H}{\partial \lambda_t}; \quad \lambda(T) = F'(\cdot) \quad x_0 = a \quad \mathbf{EQ(8)}$$

Let us now compare these conditions with their discrete-time analogues.

In both discrete- & continuous-time we have the following summary:

a. x_t, $x(t)$ - the state variable.

b. y_t, $y(t)$ - the control variable.

c. $\lambda_t, \lambda(t)$ - the adjoint or costate variable.

d. $x_{t+1} - x_t = f()$, $\dot{x} = f(\cdot)$ - the state equation/equation of motion.

e. $\partial H(\cdot)/\partial y_t = 0; \partial H(\cdot)/\partial y(t) = 0$ - the maximum condition.

f. $\lambda_{t+1} - \lambda_t = -\partial H(\cdot)/\partial x_t; \dot{\lambda} = -\partial H(\cdot)/\partial x(t)$
-The adjoint equation

Alternative terminal conditions may be considered. For example

I. Suppose x(T) = b is specified => the last term in (6) disappears (because)x(T) = 0) so that the last equation in (7) is no longer valid.

ii. If terminal time is free we must have $\partial L/\partial T = 0$ implying that $H(T) = H(x(T), y(T), \lambda(T), T) = 0 \quad \mathbf{EQ(9)}$

Equation 9 along with $F'(\cdot) = \lambda(T)$ are known as the transversality conditions

Continuous Time and the Maximum Principle - 3

The following set of equations are known as the maximum principle:

$$\frac{\partial H(\cdot)}{\partial y(t)} = 0; \quad \dot{\lambda} = -\frac{\partial H(\cdot)}{\partial x(t)}; \quad \lambda(T) = F'(\cdot) \quad H(T) = H(x(T), y(T), \lambda(T), T) = 0 \quad \mathbf{EQ(10)}$$

For an economic interpretation of $\lambda(T)$ define the maximized value function as:

$$J(x,t) = \max_{\{y(t)\}} \int_t^T V(x(t), y(t), t)\,dt \qquad \mathbf{EQ(11)}$$

for $\dot{x} = f(\cdot), y(t) \varepsilon Y,$ and *x*(*t*) = *x* (given), we can then show that, for the optimal solution $\lambda(t) = \partial J / \partial x$

Thus, the shadow price is equal to the marginal value of the state variable at time t.

The Hamiltonian thus is interpreted as the total rate of increase in the value of assets where its two terms are:

1st term = *V*(A)*,* is the flow of net returns at instant *t* while

2nd term = 8(*t*)*f*(A), is the increase in the value of the stock, *x.*

The Choice of a Discount Rate

Discounting is technique for calculating the *PV* of future flows (say net income). If *t* is discrete the PV of future net incomes N_t, t = 0,1,2,… , *T* is

$$N = \sum_{t=0}^{T} \frac{N_t}{(1+\delta)^t} = \sum_{t=0}^{T} \rho^t N_t \quad \mathbf{EQ(1)}$$

where:

$$\rho = 1/(1+\delta) = \text{the discount factor} \;\&\; \text{ä} = \text{the discount rate}$$

If t is continuous the PV of net incomes N(t), 0 # t # T is

$$N = \int_0^T N(t) e^{-rt} dt \quad \mathbf{EQ(2)}$$

e^{-rt} is the continuous discount factor and *r* is the continuous discount rate. If the time units are the same

$$e^{-r} = \frac{1}{1+\delta} \rightarrow r = \ln(1+\delta)$$

For example a 10% discount rate compounded annually is equivalent to a continuous rate of 9.53%. Other compounding periods may be treated with a similar calculation.

Note: for varying discount rates the formula will be different , but it can be shown it is the same as the previous ones if we assume the same discount rate.

Consider, for example, a discrete-time problem with a present value (PV) objective function:

$$\max \sum_{t=0}^{T-1} \rho^t V(x_t, y_t) + \rho^T F(x_T)$$

$$\text{subject to } x_{t+1} - x_t = f(x_t, y_t) \qquad \mathbf{EQ(3)}$$

$$x_0 = a \text{ given}$$

The Choice of a Discount Rate - 1

Now define a corresponding Lagrangian as:

$$L = \sum_{t=0}^{T-1} \rho^t \left\{ V(\cdot) + \rho\lambda_{t+1}\left(x_t + f(\cdot) - x_{t+1}\right)\right\} + \rho^T F\left(x_T\right) \quad \mathbf{EQ(4)}$$

λ_{t+1} is the value of an additional unit of x_{t+1} at period t+1 and must be pre-multiplied by the discount factor whereas $V_t = V(x_t, y_t)$ represents a value in period *t* and is not discounted.

The expression in {.} is a value from the perspective of period t and is discounted by ρ^t

The discrete-time *current value* Hamiltonian is

$$\overline{H}\left(x_t, y_t, \lambda_{t+1}\right) = V(\cdot) + \rho\lambda_{t+1} f(\cdot) \quad \mathbf{EQ}(5)$$

The corresponding first-order conditions are:

$$\frac{\partial H(\cdot)}{\partial y_t} = 0$$

$$\rho\lambda_{t+1} - \lambda_t = -\frac{\partial H(\cdot)}{\partial x_t} \qquad x_{t+1} - x_t = \frac{\partial H(\cdot)}{\partial\left(\rho\lambda_{t+1}\right)} \qquad \mathbf{EQ(6)}$$

$$\lambda_T = F'(\cdot) \qquad x_0 = a$$

We say "current value" since the Hamiltonian represents a value from the perspective of period *t.*

Compare the FOC with discounting to those without discounting: note the discount factor which pre-multiplies λ_{t+1}

The Choice of a Discount Rate - 2

If *t* were infinite the problem becomes:

$$\max \sum_{t=0}^{\infty} \rho^t V(x_t, y_t) \quad \text{subject to } x_{t+1} - x_t = f(x_t, y_t) \quad \textbf{EQ}(7)$$

$$x_0 = a \text{ given}$$

With the same current value Hamiltonian as earlier, the first-order conditions will be:

$$\frac{\partial V(\cdot)}{\partial y_t} + \rho\lambda_{t+1}\frac{\partial f(\cdot)}{\partial y_t} = 0 \qquad \text{a}$$

$$\rho\lambda_{t+1} - \lambda_t = -\frac{\partial V(\cdot)}{\partial x_t} - \rho\lambda_{t+1}\frac{\partial f(\cdot)}{\partial x_t} \qquad \text{b} \quad \textbf{EQ}(\mathbf{8})$$

$$x_{t+1} - x_t = f(\cdot) \qquad \text{c}$$

If a steady-state is reachable from x_o=a, evaluating 8a implies:

$$\lambda = -(1/\rho)\left[V_y / f_y\right] = -(1+\delta)\left[V_y / f_y\right] \quad \textbf{EQ}(\mathbf{9})$$

Substitute this into 8b and isolate δ on the RHS to get:

$$\delta = \left[f_x - (V_x / V_y) f_y\right] \qquad \textbf{EQ}(\mathbf{10})$$

Which is a fundamental result to models of renewable resources and is given a capital-theoretic interpretation.

Together with Equation 8c which implies f(A) = 0 when evaluated at steady state, we obtain a two equation system that may be solved for the steady-state optimum (x*, y*).

The Choice of a Discount Rate - 3

Discounting in the continuous-time model takes the following form:

$$\max \int_0^T V(x(t), y(t)) e^{-\delta t} dt + F(x(T)) e^{-\delta T}$$
$$\text{subject to } \dot{x} = f(x(t), y(t)) \qquad \textbf{EQ(1)}$$
$$x(0) = a \text{ given}$$

The Hamiltonian for this problem is:

$$H = V(\cdot) e^{-\delta t} + \lambda(t) f(\cdot) \qquad \textbf{EQ(2)}$$

The current value Hamiltonian is defined as:

$$\overline{H} = H(\cdot) e^{\delta t} = V(\cdot) + \mu(t) f(\cdot) \qquad \textbf{EQ(3)}$$

Where: $\mu(t) = e^{\delta t} \lambda(t)$ **EQ(4)**

The first-order conditions of the Hamiltonian require, in part, that:

$$\frac{\partial H(\cdot)}{\partial y(t)} = \frac{\partial V(\cdot)}{\partial y(t)} e^{-\delta t} + \lambda(t) \frac{\partial f(\cdot)}{\partial y(t)} = 0 \qquad \textbf{EQ(6)}$$

$$\dot{\lambda} = -\frac{\partial V(\cdot)}{\partial x(t)} e^{-\delta t} - \lambda(t) \frac{\partial f(\cdot)}{\partial x(t)} \qquad \textbf{EQ(7)}$$

The Choice of a Discount Rate - 4

From the definition for $\mu(t)$, we note $\lambda(t)=e^{-\delta\tau}\mu(t)$ and:

$$\frac{\partial\lambda(t)}{\partial t} \equiv \dot{\lambda} = -\delta e^{-\delta t}\mu(t) + e^{-\delta t}\dot{\mu} \quad \mathbf{EQ(7)}$$

Equations 5 and 6 may be rewritten in terms of $\mu(t) \& \dot{\mu}$ so that:

$$\frac{\partial V(\cdot)}{\partial y(t)} + \mu(t)\frac{\partial f(\cdot)}{\partial y(t)} = 0 \quad \mathbf{EQ(8)}$$

$$\dot{\mu} = -\frac{\partial V(\cdot)}{\partial x(t)} + \mu(t)\left(\delta - \frac{\partial f(\cdot)}{\partial x(t)}\right) \quad \mathbf{EQ(9)}$$

$\lambda(t) \& \mu(t)$ can be seen as present and reflecting the current value shadow prices where:

$\lambda(t)$ is the imputed value of an incremental unit in x(t) from the perspective of t-0, while:

$\mu(t)$ is the value of an additional unit of x(t) at instant t.

The complete first-order conditions, expressed in terms of the current-value Hamiltonian, can be stated as:

$$\frac{\partial\overline{H}(\cdot)}{\partial y(t)} = 0$$

$$\dot{\mu} - \delta\mu(t) = -\frac{\partial\overline{H}(\cdot)}{\partial x(t)} \qquad \dot{x} = \frac{\partial\overline{H}(\cdot)}{\partial\mu(t)} \quad \mathbf{EQ(10)}$$

$$\mu(T) = F'(\cdot) \qquad x(0) = a$$

The Choice of a Discount Rate -5

The infinite horizon problem with discounting now becomes:

$$\max \int_0^{\infty} V(\cdot) e^{-\delta t} dt$$
$$\text{subject to } \dot{x} = f(\cdot) \qquad \mathbf{EQ(11)}$$
$$x(0) = a \text{ given}$$

If a steady-state is reachable from x(0)=a, the current value Hamiltonian remains unchanged. Evaluating equation 8 in the steady-state implies:

$$\mu = -(V_y / f_y) \qquad \mathbf{EQ(12)}$$

Evaluating equation 9 in the steady-state $(\dot{\mu} = 0)$ one substitutes the expression for μ and isolating δ on the right-hand side yields:

$$\delta = (f_x - (V_x / V_y) f_y) \qquad \mathbf{EQ(13)}$$

Note that while discrete- and continuous-time analogues will typically produce identical expressions for steady state they may be subject to different dynamic behaviour.