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CRESH PRODUCTION FUNCTIONS

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The paper defines and analyzes a functional form for a one-output, many-factors production function, which is homothetic (or homogeneous), and exhibits CRES; that is, its ES (Allen-Uzawa elasticities of substitution) vary along isoquants and differ as between pairs of factors, but the ES stand in fixed ratios everywhere.

Given data on factor prices, quantities, and output, and assuming competitive cost-minimization, the parameters of CRESH are estimable from a system of log-linear equations, each containing at most three independent variables.

The CES function, as well as its limiting forms (the Cobb-Douglas ($\sigma = 1$), Leontief ($\sigma = 0$), and linear ($\sigma = \infty$) functions) are special cases of CRESH. The Mukerji CRES function has an identical unit-isoquant surface, but it is not homothetic. Appendix A analyzes the Mukerji function. Appendix B derives the (implicit) CRESH cost function.

SINCE THE BEGINNING of this decade, when the famous SMAC-CES production function [**2**] was presented and applied, a continuous search has been made for multifactor production functions with relatively simple and manageable properties. Uzawa [**23**] generalized the CES to n factors and showed that no further generalizations were possible, if the Allen-Uzawa elasticities were to remain constant. This was quite disappointing, since his function implied equality of elasticities of substitution (ES) for different pairs of factors in a group, and unitary ES for pairs of factors of different groups. McFadden [**14**] proved that, if other definitions of ES were used, their constancy implied even more stringent restrictions on the production function. Sato's [**21**] two-level CES function provided a reasonable generalization, by using a CES function among composite-goods, each of which was a CES combination of several factors. This construction, however, did not lead to any simple features for individual pairwise ES, and proved difficult to estimate. Others shared in similar results and some other modifications [**5**, **16**, **19**].

In 1963, Mrs. Mukerji [**15**] presented a function with different, but variable, ES, which had the desirable property of constant ratios of ES (CRES). Dhrymes and Kurz [**6**] applied, apparently independently, a similar function to their electricity study in 1964. This function, however, is not homogeneous or homothetic, so that individual ES vary with output as well as with factor combinations. The "expansion lines" (for given factor prices) of this function are curved in a predetermined way, and in many cases, in an undesirable way,[2] restricted by the form of the function:

$$(1) \qquad Y = [\sum D_i x_i^{d_i}]^{1/d}.$$

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[2] That is, the expansion elasticity and the substitution elasticity are both proportional to the same parameter $a_i = 1/(1 - d_i)$. In the special case $d_1 > 1$, all the factors $x_2, \ldots, x_n$ are inferior, under competitive profit maximization. See Appendix A.

In addition, apparently no one analyzed carefully the $(\{D_i\}, \{d_i\}, d)$ parameter combinations and the factor combinations $\{x_i\}$, permissible under cost minimization or profit maximization (i.e., quasi-concavity or concavity) for the function (1), and thus many interesting possibilities, as well as many peculiarities, escaped notice.[3]

Following the tradition of proving impossibility of further generalizations, Gorman [**8**] has shown in 1965 that if constant ratios of (Allen-Uzawa) ES are imposed, then the technology implied by the production function is either of "Uzawa type," or of the "Mukerji type," meaning that isoproduct surfaces are given by

$$(2) \qquad \sum D_i x_i^{d_i} = A$$

for a given output. Gorman has added, almost parenthetically, that CRES generalizations of the Mukerji function could result by having the parameters D_i and d_i as functions of output Y, rather than constants.[4] This would alter the pattern of change of individual ES with Y and the curvature of the expansion lines, but will not remove the restrictions on combinations of ES permissible under (2)—while preserving the CRES property throughout. Gorman, however, wrongly claimed that ES could not be negative,[5] and apparently failed to notice the richness of the class of functions defined by (2), where the parameters vary with output. As an unfortunate result of these developments, it has been widely believed that the class of CRES functions is relatively limited in flexibility. It has also been implied (wrongly) that this "Mukerji technology" necessarily leads to non-homogeneous or non-homothetic production functions.[6]

In this paper a class of multi-factor CRES production functions which are homothetic or homogeneous (hence CRESH) is presented and analyzed. The restrictions on its domain and on the parameters implied by quasi-concavity (or cost minimization under competitive factor markets) are fully examined in Section 1. These apply to Mukerji's function (1) as well, since the unit isoquant surfaces of the two types of functions are identical.[7] The additional restrictions implied by concavity (or competition in the product market) are also indicated, while Appendix A analyzes these restrictions for the function (1) and shows some of its peculiar features.

In addition, some possibilities for statistical estimation of this function, as well as for testing it against CES and other functional forms, are indicated briefly in Section 2. It is shown below that under cost minimization in competitive factor

[3] See Cook [**4**], Dhrymes and Kurz [**6**], Mukerji [**15**], and the analysis in Appendix A.

[4] Actually, $d_i(Y)$ must be of the form $d_i = 1 - (1/a_i)\theta(Y)$, where the a_i's are constant. See Gorman [**8**, p. 218]. The right-hand side $A = \theta(Y)$ in Gorman may be absorbed in the functions $D_i(Y)$. For the function (1), $A = 1$, $\theta(Y) \equiv 1$, and $D_i(Y) = D_i Y^{-d}$.

[5] This was pointed out by Ramanujam [**18**].

[6] Solow [**22**, p. 46] implies both of these assertions.

[7] However, the additional restriction $D_i > 0$, all i, is required for the function (1) to exist (with a non-negative, finite output) for all $\underset{\sim}{x} \gg \underset{\sim}{0}$. ($\underset{\sim}{x} \gg \underset{\sim}{0}$ means $x_i > 0$, all i; $\underset{\sim}{x} \geqslant \underset{\sim}{0}$ denotes $x_i \geqslant 0$, all i; and $\underset{\sim}{x} > \underset{\sim}{0}$ denotes $\underset{\sim}{x} \geqslant \underset{\sim}{0}$ but $\underset{\sim}{x} \neq \underset{\sim}{0}$.)

markets, CRESH yields the following system of log-linear equations:

$$\text{(24)}\qquad \log x_i = b_{i1}\log x_1 + b_{iy}\log h(y) + b_i \log(p_i/p_1) + b_{i0} \qquad (i = 2,\ldots,n),$$

which provides a convenient basis (given additional specifications for the stochastic elements) for estimation and testing. It is our intention to carry out shortly some empirical investigations along these lines, where a more thorough econometric analysis will be presented.

Finally, a few concluding remarks (Section 3) refer to the potential of this class of functions as well as to its limitations, and to some possible generalizations and modifications. Appendix B derives the cost function corresponding to CRESH.

1. THE CRESH FUNCTION

Let $\underline{x} = (x_1, \ldots, x_n)$ denote a vector of n inputs, and Y output. The production function $Y = f(\underline{x})$ is defined implicitly by the following equation:

$$\text{(3)}\qquad F(Y, \underline{x}) = \sum_{i=1}^{n} D_i[x_i/h(Y)]^{d_i} - 1 \equiv 0,$$

for $x_i > 0$, $0 \leqslant Y \leqslant \bar{Y} \leqslant +\infty$, where $h(Y)$ is continuously differentiable with $h(0) = 0$, $h(\bar{Y}) = +\infty$, and $h'(Y) > 0$. ($\bar{Y}$ denotes maximum producible output.) Also, $F(0, \underline{0}) = 0$.

Equation (3) yields a unique, continuously differentiable production function $Y = f(\underline{x})$, if and only if $\max_i D_i > 0$, and $\{D_i\, d_i\}$ are of the same sign for all i.[8] This function $Y = f(\underline{x})$ is clearly homothetic, satisfying $h[f(t\underline{x})] = th[f(\underline{x})]$. That is, $h(Y)$ is linear homogeneous with respect to $\underline{x}$.[9] If one chooses $h(Y) = Y^{1/\mu}$, then $f(\underline{x})$ is homogeneous of degree μ, i.e., $f(t\underline{x}) = t^{\mu} f(\underline{x})$. The constant returns to scale (CRS) case is $h(Y) = Y$. For the case $h(Y_0) = 1$, the isoproduct surface is clearly identical with that of the Mukerji function (1) for $Y = 1$. But for other levels of output, $f(\underline{x}) = Y_0$ is expanded uniformly (or homothetically), and differs from (1).

Assuming a set of n (shadow or competitive) factor prices $\underline{p}$, we can analyze the first and second order conditions for cost minimization, which are equivalent to the conditions for quasi-concavity of $f(\underline{x})$. For minimizing cost $C = \Sigma\, p_i x_i$, subject to (3), form the Lagrangian

$$\text{(4)}\qquad \bar{L} = \bar{L}(\underline{x}; Y, \underline{p}) = C - \lambda F = \sum_k p_k x_k - \lambda\left\{\sum_k D_k \left[\frac{x_k}{h(Y)}\right]^{d_k} - 1\right\},$$

[8] It may be shown (details of the proof are omitted here), that under these conditions the function F satisfies the requirements of the implicit function theorem for yielding a unique, monotone $y = f(\underline{x})$ for all $\underline{x} \gg \underline{0}$: $F(0, \underline{x})$ and $F(\bar{Y}, \underline{x})$ always bracket zero, and $F_Y F_{x_i} < 0$. (See Hadley [11, p. 47].)

The case $d_j = 0$ for some j may be included, if the corresponding term is replaced by $B_j \log[x_j/h(Y)]$, and B_j is of the same sign as all $D_i d_i$. In addition, one could add "limitational factors" in the form: $x_{n+k} = \gamma_k h(Y)$ without affecting the results. See Gorman [8, p. 218], and the discussion of these cases (as well as the case $d_j = 1$) below.

[9] This is in accordance with the accepted definitions of homotheticity. The definition given by Lancaster [12, p. 334], $f(t\underline{x}) = \theta(t) \cdot f(\underline{x})$, is in error since it corresponds to homogeneous functions only (of degree $\theta'(1)$). See Hanoch [10].

where λ is a Lagrange multiplier, and is a linear homogeneous function of all prices $\underline{p}$.

In order to obtain the results with relative algebraic ease, we carry out the following transformations of variables and parameters:

$$(5) \qquad \log x_i = z_i, \qquad \log h(Y) = y, \qquad \log p_i = q_i, \qquad \log|\lambda| = \Lambda,$$
$$\log|D_i d_i| = c_i, \qquad \frac{1}{1-d_i} = a_i.$$

The Lagrangian function (4) is now transformed:

$$(6) \qquad L(\underline{z}; y, \underline{q}) = \sum_k \exp(q_k + z_k) - \lambda\left\{\sum_k D_k \exp[d_k(z_k - y)] - 1\right\}.$$

The first order conditions for minimum C are, in these terms,

$$(7) \qquad \frac{\partial L}{\partial z_i} = \exp(q_i + z_i) - \lambda D_i d_i \exp\{(d_i(z_i - y)\} = 0 \qquad (i = 1, \dots, n).$$

From these we infer that all $D_i d_i$ have the same sign as λ, since $\lambda D_i d_i > 0$. Note that neither $D_i > 0$ nor $D_i d_i > 0$ is necessary. Equations (7) yield (using the definitions (5)):

$$(8) \qquad z_i = a_i(\Lambda - q_i - d_i y + c_i).$$

If the system of $(n + 1)$ equations (3) and (8) is solved for $\Lambda(\underline{q}, y)$, and the result substituted in (8), then equations (8) give the factor demand functions, at a given y, where $x_i(\underline{p}, Y)$ are homogeneous of degree 0 in $\underline{p}$.

From (8) the price elasticities of factor demands are directly derived:

$$(9) \qquad \begin{aligned} \eta_{ii} &= \frac{\partial z_i}{\partial q_i} = a_i \frac{\partial \Lambda}{\partial q_i} - a_i, \\ \eta_{ij} &= \frac{\partial z_i}{\partial q_j} = a_i \frac{\partial \Lambda}{\partial q_j}, \qquad i \neq j \end{aligned} \qquad (i = 1, \dots, n).$$

To solve for $(\partial\Lambda/\partial q_j)$, use the restriction

$$(10) \qquad \sum_k \eta_{kj} s_k = 0,$$ [10]

where $s_i = (p_i x_i / \Sigma\, p_k x_k)$ is the share of the ith factor in total cost (since $Y = f(\underline{x})$ is homothetic, s_i is clearly independent of Y): $\Sigma_k \eta_{kj} s_k = (\partial\Lambda/\partial q_j)\Sigma\, s_k a_k - s_j a_j = 0$; hence

$$(11) \qquad \frac{\partial \Lambda}{\partial q_j} = \frac{s_j a_j}{\sum s_k a_k}.$$

[10] Allen [**1**, pp. 503–509].

Substituting (11) in (9) gives the elasticities of demand:

$$(12)\qquad \eta_{ii} = \frac{s_i a_i^2}{\sum s_k a_k} - a_i \qquad (i = 1, \ldots, n),$$

$$\eta_{ij} = \frac{s_j a_j a_i}{\sum s_k a_k}, \qquad j \neq i.$$

The ES (Allen-Uzawa pairwise partial ES) are related to the demand elasticities as follows:

$$(13)\qquad \sigma_{ij} = \sigma_{ji} = \frac{1}{s_j}\eta_{ij} = \frac{a_i a_j}{\sum s_k a_k}, \qquad i \neq k,$$

and the "own elasticity" terms[11] are, similarly,

$$(14)\qquad \sigma_{ii} = \frac{1}{s_i}\eta_{ii} = \frac{a_i^2}{\sum s_k a_k} - \frac{a_i}{s_i}.$$

The CRES property is thus clear, since all σ_{ij} vary proportionately, with the common factor of proportionality being $1/\Sigma\, s_k a_k = 1/\bar{a}$, where $\bar{a}$ is a weighted average of $\{a_i\}$, with the factor shares $\{s_i\}$ as weights.

To analyze the restrictions on the parameters D_i and a_i (or d_i), implied by second-order conditions for cost minimization, i.e., by quasi-concavity of $f(\underline{x})$, note that these conditions may be specified in terms of the matrix $\Sigma_n = [\sigma_{ij}]_n$ of ES: Σ_n should be negative semi-definite.[12] Thus we require

$$(15)\qquad (-1)^m |\Sigma_m| > 0 \qquad (m = 1, \ldots, n-1)$$

and

$$|\Sigma_n| = 0,$$

where $|\Sigma_m|$ is the principal minor determinant of order m of Σ_n. Substituting (13) and (14) in (15) yields, after some manipulations,

$$(16)\qquad (-1)^m |\Sigma_m| = \left(\prod_{i=1}^{m} \frac{a_i^2}{s_i}\right)\left(\prod_{i=1}^{m} \frac{1}{a_i}\right)\left(1 - \sum_{i=1}^{m} \frac{s_i a_i}{\bar{a}}\right) \qquad (m = 1, \ldots, n).$$

Clearly, $|\Sigma_n| = 0$ (since $\bar{a} = \Sigma_{i=1}^n s_i a_i$), which is a natural result of the zero homogeneity in $\underline{p}$ of the factor demands. If all prices change in the same proportion, no change occurs, and $|\Sigma_n| = 0$; hence Σ_n is semi-definite.

For $m < n$, however, equation (16) may be shown to be equivalent to the following conditions for a (regular interior) minimum cost:

$$(17)\qquad \left(\prod_{i=1}^{m} a_i\right) \sum_{i=1}^{m} s_i a_i > 0 \qquad (m = 1, \ldots, n).$$

[11] Allen [**1**, pp. 503–509].

[12] See Samuelson [**20**, p. 78] and Allen [**1**, p. 505].

Conditions (17) must apply to the Mukerji function (1) as well, since the unit isoquants coincide, and output is held constant.

There are several immediate interesting implications to be drawn from (17). (i) If all $a_i > 0$, the condition holds everywhere for $\underset{\sim}{x} \gg \underset{\sim}{0}$. (In particular, the signs of D_i are immaterial, except that they may not be all negative by (3).) (ii) If a_1 (say) is negative, and $a_i > 0$ for $i = 2, \ldots, n$, condition (17) may still hold locally if $\Sigma_{i=1}^{n} s_i a_i < 0$, or

$$\text{(18)} \qquad -s_1 a_1 > \sum_{i=2}^{n} s_i a_i > 0.$$

(iii) Not more than one a_i may be negative. Suppose $a_1 < 0$ and $a_2 < 0$. But this implies $a_1 a_2 (s_1 a_1 + s_2 a_2) < 0$, which contradicts (17) for the case $m = 2$.

We thus have two general cases.

CASE I: All $a_i > 0$. Hence by (13) all ES are positive, and all factors are substitutes. For any $a_i > 1$, we have $0 < d_i < 1$ and for any $0 < a_i < 1$, $d_i < 0$. Remember also that all $D_i d_i$ must have the same sign, and $\max_i D_i > 0$; i.e., if either all $a_i > 1$, or all $0 < a_i < 1$, then all $D_i > 0$. But in other cases, some D_i may be positive and some negative, corresponding to $a_i > 1$ or $0 < a_i < 1$, respectively, *or vice versa*.

CASE II: $a_1 < 0$; then $a_i > 0$ $(i = 2, \ldots, n)$ by (ii) and (iii) above. This implies also $d_1 > 1$, and $\bar{a} = \Sigma_{i=1}^{n} s_i a_i < 0$ by (18). Using (13), we get $\sigma_{1j} = (a_1 a_j)/\bar{a} > 0$ and $\sigma_{jk} < 0$ for $j > 1$, $k > 1$. Thus, in this case factor x_1 (say) is a substitute for all other factors, while the factors $x_2, \ldots, x_n$ form a group of *complements*, with negative ES for any pair in the group. However, this case is not valid globally for all $\underset{\sim}{x} \gg \underset{\sim}{0}$. The restriction (18) implies, for $a_1 < 0$,

$$-1 < \sum_{i=2}^{n} \frac{x_i p_i a_i}{x_1 p_1 a_1} < 0.$$

Using (7) we get

$$\frac{x_i p_i}{x_1 p_1} = \frac{D_i d_i}{D_1 d_1} [x_i/h(Y)]^{d_i} [x_1/h(Y)]^{-d_1},$$

and hence, substituting in the above and solving for $[x_1/h(Y)]$, the following restriction is obtained:

$$\text{(19)} \qquad x_1/h(Y) > \left\{ \sum_{2}^{n} \frac{D_i d_i a_i}{D_1 d_1 |a_1|} [x_i/h(Y)]^{d_i} \right\}^{1/d_1}.$$

The domain where (19) holds is a cone with large relative values of x_1. At smaller x_1 values, the isoquant surfaces are not convex.[13]

[13] Thus $d_i \leqslant 1$ $(a_i \geqslant 0)$ are necessary and sufficient conditions for global quasi-concavity for all $\underset{\sim}{x} \gg \underset{\sim}{0}$ (in addition to the specified conditions on D_i).

We now consider the special case (see Footnote 8 above) $d_i = 0$ or $a_i = 1$, and also the limiting cases $d_i \to -\infty$ or $a_i = 0$ (limitational factors), and $d_i = 1$ or $a_i \to \infty$ (perfect substitutes). It may be established now that if any term in (3) is replaced by $B_j \log(x_j/h(Y))$, all the above results hold as if the corresponding a_j is equal to 1 (or $d_j = 0$) and $D_j d_j = B_j$. The corresponding term in equation (6), $D_j \exp[d_j(z_j - y)]$, is replaced by $B_j(z_j - y)$, and equation (7) for this factor becomes

(7)′ $$\frac{\partial L}{\partial z_j} = \exp(q_j + z_j) - \lambda B_j,$$

and equation (8) becomes

(8)′ $$z_j = \Lambda - q_j + c_j, \quad \text{where} \quad c_j = \log B_j.$$

From (8) on, all the results are valid for this generalization, with $d_j = 0$ and $a_j = 1$. (If, however, all $a_j = 1$, the function $f(\underset{\sim}{x})$ obviously reduces to the Cobb-Douglas case, or a monotonic transformation $h(f)$ of it, and all ES are $= 1$.)

Similarly, the case of limitational factors, $x_{n+k} = g_k(Y) = \gamma_k \cdot h(Y)$ (to preserve homotheticity in all factors), may also be integrated into the above results, if one puts $a_{n+k} = 0$ for such factors. The ES of any such factor is then zero by (13). (If all $a_i = 0$, the production function degenerates into the constant-coefficients Leontief type; cf. Gorman [**8**].)

The case $d_m = 1$ or $a_m \to \infty$ is integrated here by noting that the corresponding factor enters linearly into F of (3); hence it is a perfect substitute for the "composite factor" made up by all the other factors. The expression z_m is now cancelled out from the equation corresponding to (7), and the marginal conditions can not be satisfied. The result is a corner solution, where either only x_m is employed at a given output, or no x_m at all. Formally, the elasticity of substitution for this factor with any other factor x_i (where $d_i \neq 1$) yields

$$\sigma_{mi} = \lim_{a_m \to \infty} \frac{a_m a_i}{\sum s_k a_k} = \lim_{a_m \to \infty} \frac{a_i}{s_m}$$

(taking ratio of derivatives), and is either a_i, if only x_m is employed and $s_m = 1$; or ∞, if no x_m is employed and $s_m = 0$. (A function with all $d_m = 1$ has linear isoproduct surfaces with the obvious results.)

Finally, if all a_i are equal, one gets back, of course, to the CES technology (meaning any positively monotonic transformation $h(Y)$ of $Y = f(\underset{\sim}{x})$ which is CES).

This completes the analysis and classification of parameter values and input combinations which are consistent with cost minimization, or quasi-concavity. If one adds the assumption of profit maximization under a competitive product market, however, or a given constant shadow price for output, the production function must be concave throughout the range of potential optimum points. This, in turn, is equivalent to a non-decreasing long-run average cost curve in the relevant range. Since the CRESH function defined in (3) is homothetic, however, its cost function is separable into a product of two functions:

(20) $$c(\underline{p}, Y) = g(\underline{p}) \cdot h(Y),$$

where $g(\underline{p})$ is the "unit cost" function[14] (i.e., the cost of producing Y_0, where $h(Y_0) = 1$), which is independent of Y, and is zero homogeneous in factor prices; and $h(Y)$ is independent of $\underline{p}$, and is linear-homogeneous in $\underline{x}$. The structure (20) implies, then,

(21) $$\frac{\partial c}{\partial Y} - \frac{c}{Y} = g(\underline{p}) \cdot \left(h'(Y) - \frac{h(Y)}{Y}\right) \geqslant 0.$$

Since $g(\underline{p}) > 0$, the additional restriction amounts to assuming $Yh'(Y)/h(Y) \geqslant 1$. In particular, no additional restrictions are imposed on the domain of $\underline{x}$ or on the parameters of (3). In the homogeneous case, $h(Y) = Y^{1/\mu}$, so that (21) implies $0 < \mu \leqslant 1$.

The preceding analysis does not apply, however, to the non-homothetic function (1). Appendix A shows that in addition to (17) the concavity of the function (1) imposes additional restrictions on the parameter d and on the domain of variation of $\underline{x}$. It is also shown there that in the case corresponding to Case II above (i.e., a_1 negative), all factors except x_1 (say) are inferior factors, such that an *increase* in demand (factor prices constant) gives rise to a *reduction* in all inputs[15] except x_1. Both these features stress the undesirable properties of the CRES function (1) that result from its particular form—totally independently of its "technology," meaning the shape of a given isoproduct surface, which is identical with that of our CRESH function. The way these surfaces change with a changing level of output creates the unwarranted features of the function (1).

2. ESTIMATION AND TESTING OF HYPOTHESES

Any empirical application of the CRESH function requires, first of all, a particular specification of the function $h(Y)$ which enters the definition (3). As indicated above, this function determines the behavior of the cost function with respect to output. Hence if one feels that the model which generated the relevant data justifies the assumption of homogeneity, the specification $h_1(Y) = Y^{1/\mu}$ could be chosen. For many types of models and data (particularly for time-series aggregate models) it is both reasonable and customary to assume constant returns to scale, with a perfectly elastic unit cost curve, implying $h_2(Y) = Y$. For other types of data generation models (e.g., for cross-sectional studies of individual firms in competitive factor markets), one would like to have an S-shaped total cost curve, which yields a U-shaped average cost curve. There are many functional forms in this general class. A particularly convenient one is, e.g., $h_3(Y) = \exp(Y/Y_0)$, which has

[14] In Appendix B the function $g(\underline{p})$ is derived and related to $\lambda(\underline{p})$ of equation (4) above.

[15] An equivalent result is that marginal cost decreases with an increase in the price of any inferior factor (at a given output), and hence that the quantity of output supplied under competition increases with p_j ($j \neq 1$). Cf. Samuelson [**20**, p. 66] and Bear [**3**].

a minimum average cost at $Y = Y_0$. The estimation equations presented in this section are based on these three specific functions h_1, h_2, or h_3. The modifications required for other types of $h(Y)$ would be relatively self-evident.

First, applying equation (8) above to $i = 1$, and solving for Λ gives

$$\Lambda = (z_1/a_1) + q_1 + d_1 y - c_1. \tag{22}$$

Substitution of (22) in (8) and rearrangement yield the following system of $(n - 1)$ equations:

$$z_i = \frac{a_i}{a_1} z_1 + \left(1 - \frac{a_i}{a_1}\right) y - a_i(q_i - q_1) + a_i(c_i - c_1) \qquad (i = 2, \ldots, n). \tag{23}$$

In terms of the original variables (23) becomes

$$\log x_i = b_{i1} \log x_1 + b_{iy} \log h(Y) + b_i \log \frac{p_i}{p_1} + b_{i0}, \tag{24}$$

where $b_{i1} = (a_i/a_1)$, $b_{iy} = 1 - (a_i/a_1)$, $b_i = -a_i$, and $b_{i0} = a_i \log (D_i d_i/D_1 d_1)$, $i = 2, \ldots, n$.

Let us start with the general homogeneous case $h(Y) = Y^{1/\mu}$. In this case, the term $b_{iy} \log h(Y)$ in (24) is replaced by $B^1_{iy} \log Y$, where $B^1_{iy} = (1/\mu)(1 - (a_i/a_1))$. The coefficients are now subject to the following linear restrictions:

$$a_1 b_{i1} + b_i = 0 \quad \text{and} \quad b_{i1} + \mu B^1_{iy} = 1 \qquad (i = 2, \ldots, n), \tag{25}$$

where a_1 and μ are constant for all i.

If the coefficients of any equation in (24) are known, one can clearly solve for the parameters μ, a_i (or d_i), a_1 (or d_1), and D_i/D_1. If consistent estimates can be found for these coefficients, under any statistical model, then consistent estimates of the above are also given. Since the level of individual D_i depends on units of measurement, it can only be computed (or estimated) directly from the production function relation (3) (with not much economic contents, however).[16]

For the CRS case, $h_2(Y) = Y$, equation (24) may be transformed as follows:

$$\begin{aligned} \log\left(\frac{x_i}{Y}\right) &= b_{i1} \log\left(\frac{x_1}{Y}\right) + b_i \log\left(\frac{p_i}{p_1}\right) + b_{i0} \\ &= b_i\left[\log\left(\frac{p_i}{p_1}\right) - \frac{1}{a_1}\log\left(\frac{x_1}{Y}\right)\right] + b_{i0} \qquad (i = 2, \ldots, n). \end{aligned} \tag{26}$$

Several alternative formulations may be derived from equation (26) (or (23)). The following is a noteworthy example:

$$\log\frac{x_i}{x_1} = a_i \log\frac{p_1}{p_i} + \frac{a_1 - a_i}{a_1}\log\frac{Y}{x_1} + a_i(c_i - c_1). \tag{27}$$

[16] Sometimes it is convenient to define $D_i = A\delta_i$, where δ_i are subject to a "normalization scheme," e.g. $\Sigma\delta_i = 1$. The δ_i are called "distribution parameters," and they may now be derived from the given set $\{D_i/D_1\}$. Cf. Arrow et. al. [**2**], and Dhrymes and Kurz [**6**, p. 289].

If $a_i = a_1$ for some i, the second term on the right cancels out, and the equation is that given in Arrow et al. [**2**] for CES in the two-factor case. The variable Y/x_1 thus represents, in each of equations (27), the deviation from constant ES,[17] for any pair of factors.[18]

For $h_3(Y) = \exp(Y/Y_0)$, $\log h_3(Y) = Y/Y_0$, a similar procedure yields

$$\log x_i = b_{i1} \log x_1 + B_{iy}^3 Y + b_i \log\left(\frac{p_i}{p_1}\right) + b_{i0}, \tag{28}$$

where

$$B_{iy}^3 = \frac{1}{Y_0}\left(1 - \frac{a_i}{a_1}\right) \quad \text{and} \quad a_1 b_{i1} + b_i = 0 \qquad (i = 2, \ldots, n).$$

Clearly, if either $\log(p_i/p_1)$, or $\log Y$ (Y in the case of $h_3(Y)$) is to be regarded as the dependent variable in the particular specification of the model, then equations equivalent to (24), (26), or (28) may be derived from these with no difficulty, and will preserve their linear properties.

In order to simplify the following discussion, it is assumed now that the statistical models corresponding to equation (24) (and the like) consist of the given equations, with an additive error term U_i, where U_{it} have zero means, constant variance-covariance matrix, and no serial correlation. Modifications of these methods, required if additional complications (such as serial correlation of U_{it}) are present, are dealt with extensively in the econometric literature, and would best be avoided here.

The equations presented above are suitable for estimating models where all the inputs x_i are regarded as endogenous, determined by the marginal conditions for cost minimization under competitive factor markets, whereas both relative factor prices p_i/p_1 and output Y, are determined exogenously. There are two problems connected with the estimation of these equations. First, in all these equations (24), (26), and (28), a term depending on the endogenous variable x_1 appears on the right-hand side. Hence direct least squares (regression) estimates will in general be subject to simultaneity bias.[19] To get consistent estimates for the parameters, one may use the instrumental variable method in a two stage procedure, where in the first stage the appropriate (x_1) variable ($\log x_1$ or $\log(x_1/Y)$) is regressed against all the

[17] In the sense of "Direct ES (DES)" defined for substitution between two factors if all other factors are held constant. Cf. McFadden [**14**, p. 73]. (We may note here, that the DES for CRESH are given by: $D_{ij} = a_i a_j/\bar{a}_{ij}$, where $\bar{a}_{ij} = (s_i a_i + s_j a_j)/(s_i + s_j)$, and therefore the DES do not preserve constant ratios.)

[18] A somewhat similar analysis, for the two-factor CRS case, was used by Liu and Hildebrand [**13**], who added a term in (x_1/x_2) to the SMAC equation relating Y/x_1 to p_1. Bruno (see Nerlove [**17**, p. 75]) integrated the equation to get the corresponding production function. Their function is different, however, from the two-factor case of CRESH.

[19] An x_1 which is determined exogenously, e.g., capital in the short run, will not satisfy the marginal conditions, and hence cannot serve as a "numeraire" in equations like (24). In this case one has to replace x_1 by a variable (endogenous) factor, e.g., x_2, in these equations, and the number of equations in the system is reduced to $(n - 2)$. The parameters connected with x_1 are then estimable in a non-linear fashion from the production relation (3), given the estimates for all the parameters connected with the other factors $x_2, \ldots, x_n$.

exogenous variables of the system, and in the second stage the computed value $(x_1)^*$ is used as an instrumental variable for estimation of (24) or the equivalent equation. These estimates will be consistent and their efficiency will depend on the goodness of fit in the first stage [i.e., on $r^2(x_1)^* \cdot (x_1)$].[20]

The second problem arising in the estimation of these equations is due to the linear restrictions (such as (25)) placed on the parameters. One may follow two alternative routes: (i) estimate the equations by the above method, disregarding the linear restrictions, and then perform statistical tests of the model, by testing whether there exist significant deviations from the restrictions given in (25); or (ii) use a scheme such as Zellner's [**24**], adapted to the instrumental variables method, to estimate all equations simultaneously under the linear restrictions.[21] We recommend using first (i) and then, if the model is not rejected, proceeding to get more efficient estimates by using (ii).

In many applications, however, the level of output Y may not be regarded as exogenously determined. Instead, it is determined within the system, through a profit maximization mechanism, where the exogenous element is either the product price p (under competition), or a demand function $Y = Q(p)$ (e.g., under monopoly). Substituting for Y in equations (24) the exogenous element p (or a demand shift parameter) does not yield, however, equations which are linear in logs, or otherwise simple. Therefore to estimate systems such as (24) regarding Y as endogenous we recommend extending the instrumental variable methods explained above to the case of two endogenous variables.[22]

Maximum likelihood estimation of non-linear equations derived from this model (including direct, non-linear estimation of the production function relation (3) itself), are also feasible with current computer technology, for relatively small n. Their description, however, would be beyond the scope of this survey.

In what follows, we present one additional estimation procedure, which is based on approximate equations, rather than on exact relations.

Suppose that a first-order Taylor expansion of the (non-linear) function $\Lambda(\underline{q})$ of equation (8) gives a reasonably good approximation around some $(\underline{q}_0)$. We therefore assume

$$(29) \qquad \Lambda(\underline{q}) = c(\underline{q}_0) + \sum_{j=1}^{n} \frac{\partial \Lambda}{\partial q_j} q_j.$$

Substitution of $(\partial \Lambda / \partial q_j)$ from (11) into (29) and introduction of the result into equation (8) leads to the following system:

$$(30) \qquad z_i = B_{i0} - a_i(q_i - \bar{q}) - (a_i - 1)y \qquad (i = 1, \ldots, n),$$

[20] See Goldberger [7, p. 286]. The simple two-stage least squares method is not valid here, since the true reduced-form equations are not linear. However, if multicollinearity is not prohibitive, one can use second-order terms in the first stage, for computing $(x_1)^*$ with a better fit.

[21] A detailed derivation of this method is outside the scope of this paper.

[22] For more than three factors, identification is guaranteed within the system, since the number of exogenous price-ratios excluded from each equation is at least two. (The efficiency may be low, however, if the correlation of these price ratios with Y is low.) For three factors, however, an external exogenous variable is required, e.g., a fourth factor which is exogenous (see footnote 19) or the exogenous product-price in the case of a competitive product market.

where

$$\bar{q} = \frac{\sum^{n} s_i a_i q_i}{\sum^{n} s_i a_i},$$

and B_{i0} is a constant ($= a_i[c_i + c(\underline{q}_0)]$). For the homogeneous case $h_1(Y) = Y^{1/\mu}$, we get

$$\text{(31)} \qquad \log x_i = B_{i0} - a_i(\log p_i - \overline{\log p}) - \frac{1}{\mu}(a_i - 1)\log Y \qquad (i = 1, \ldots, n),$$

where $\overline{\log p} = \bar{q}$ is a weighted mean of all the (exogenous) $\log p_i$, with $s_i a_i$ as (variable) weights. However, since presumably a weighted mean is not too sensitive to weights,[23] one may try the following iterative procedure.

(i) Compute for each observation, $\overline{\log p^*} = \Sigma_i s_i \log p_i$, disregarding the differences among a_i.[24]

(ii) Estimate the system of n equations (31) by simple least squares (under the linear restriction involving the second and third coefficients), using $\overline{\log p^*}$ instead of $\overline{\log p}$.

(iii) Use the a_i estimated in (ii) to re-estimate $\overline{\log p^{**}}$ for each observation, and iterate on (ii).

(iv) One can then perform tests on the generalized residual variance to be satisfied that the second iteration (or consecutive ones) indeed improve the explanatory power of the system, and that the estimated $\{a_i\}$ tend to stabilize. Using these estimates will then provide estimates for all ES and for the returns to scale parameter μ (but not for the D_i or δ_i).

Alternatively, equations (31) may be estimated by non-linear methods to give approximate maximum likelihood estimates, e.g., by scanning over different values of $\{a_i\}$ to minimize the generalized residual variance—if n is not too large.

3. CONCLUDING REMARKS

Let us summarize briefly some of the main features of the CRESH production function (3), in comparison with other functional forms currently in common usage. Unlike Cobb-Douglas and CES (or monotonic transformations of these), the CRESH function does not have constant individual (Allen-Uzawa) ES for changing factor proportions. However, all ES vary proportionately, and presumably very gradually, since the variable element $\bar{a} = \Sigma s_i a_i$ is a weighted mean of the constant a_i's. There is no economic a priori preference, however, for constancy of ES, except for empirical indications of relative constancy in two-factor aggregate models, as explained by SMAC [**2**], and the relatively convenient estimation equations produced by CES. If one is interested in studying patterns of substitution or complementarity among three factors or more, however, as would be the case in

[23] The difference may be shown to be of a second order of magnitude, under reasonable assumptions.

[24] For testing against CES, this is the true $\bar{q}$ under the null hypothesis $a_i = a_j$, all i, j, and the procedure is therefore entirely valid at the first iteration. Both s_i and $\log p_i$ are observable magnitudes.

the current widely researched area of the interrelations among different types of labor and skills, or in distinguishing various inherently different forms of capital (e.g., structures, equipment, raw materials), the CES model stands in sharp contradiction to economic common sense as well as to the very purpose of such studies. It is the *differences* among ES which are the focus of attention, whereas the CES model assumes these differences away.

The CRESH function presented here allows for such differences, and simplifies matters by having constant relative differences of ES. Its flexibility is sufficiently extensive to encompass cases of negative ES, or complementarity (although in a rather limited fashion), as well as widely varying degrees of substitutability.

At the same time, the CRESH function is relatively manageable in terms of the estimation procedures available, such as those suggested in Section 2. Although less simple than the CES equations (or obviously, the Cobb-Douglas special case), these equations offer no insurmountable statistical or computational difficulties for many aggregate or disaggregate economic models. It remains to be verified empirically, however, that the quality of the available data justified this degree of sophistication.[25]

Comparing our function with the Mukerji function (1), the CRESH model is to be preferred for several reasons: having the same "technology" (or unit isoquant surface) gives rise to the same patterns of ES, while it is well behaved (i.e., concave) in a wider domain of factor combinations. The fact that CRESH is a homogeneous (or a homothetic) function makes it a more natural generalization of the homogeneous CES model, and more adaptable to integration into current economic theory and practice—on the micro level as well as on the aggregate level, and especially in studies of economic growth and technical change.

It may be argued that technical change, specifically of the factor-augmenting or biased variety, requires, by definition, non-homothetic production function models. Hence one could argue that the Mukerji function would be more suitable for its study. This, however, is an obvious non-sequitur. The "bias," or curvature of expansion lines inherent in that function, is predetermined and arbitrary, and bears no relation to the bias produced by an assumed pattern of technical change, except by a highly improbable coincidence. Appendix A shows that the curvature of the expansion lines of the Mukerji function is quite peculiar since it depends on the same parameters as the ES.

As a contrast to the above, the CRESH function can easily be modified and generalized to incorporate any desired pattern of technical change; in fact, the implicit form of the defining equation (3)—although somewhat inconvenient in other respects—is particularly convenient for this purpose:

[25] It has been argued in the recent literature (e.g., Nerlove [**17**, p. 56], and Griliches [**9**]) that deviations of ES from 1 are second-order magnitudes, and hence are not effectively estimable with the kind of data that is currently available. This would lead to preference for the Cobb-Douglas simple form over both the CES and the CRESH models. It seems that these arguments, however, are based on results which imposed CES behavior on data that did not satisfy it. In the more general CRESH model, this argument may no longer be valid.

(i) Any neutral, unbiased, technical change involves merely a modification of the function $h(Y)$ to make it rising with output, with accumulated investment, or with time t in a given desirable way. This obviously does not change equation (3) for a given level of $h(Y)$, and thus the isoquant surface map is invariant.

(ii) Any technical change which is "embodied" in x_i, or is x_i-augmenting, is equivalent to changing units of measurement for x_i with respect to time t or output Y. Hence it is simply incorporated in (3) by replacing the constant D_i coefficient by an appropriate function $D_i(t)$, or $D_i(Y)$.

Both these modifications will not change the *ratios* of ES among factors (since these depend only on d_i), but individual elasticities would change under (ii), for any given factor prices.

It would be outside the scope of our presentation to plunge into these matters in any detail, or to try additional generalizations.

As a final remark it is suggested that the CRESH functional form may prove to be useful as an n-commodity utility function, in cases where homotheticity (i.e., unitary elasticities with respect to income, or wealth) may reasonably be assumed, as in the Fisher-type analysis of optimization of total consumption over time. In other cases, the more general CRES model of Gorman [**8**] may be used, with estimation equations quite similar to the CRESH equations presented here.

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APPENDIX A: AN ANALYSIS OF MUKERJI'S FUNCTION

The function given in (1) is defined explicitly by

$$Y = M(\underline{x}) = \left(\sum_{i=1}^{n} D_i x_i^{d_i}\right)^{1/d}. \tag{A.1}$$

Since for unit output ($Y = 1$) the equation of the unit isoquant surface $M[\underline{x}(1)] = 1$ is identical with the "unit" isoquant ($h(Y) = 1$) of the CRESH function (3), it is evident that cost minimization at this output has exactly the same implications for the domain of $\underline{x}(1)$ and the valid $\{D_i, d_i\}$ parameter combinations as the CRESH function, and need not be re-examined here. However, the restrictions on the parameter d, the domain of $\underline{x}$ for $Y \neq 1$, and the restrictions implied by concavity of M (or profit maximization under competition) require a separate analysis.

It can be verified that the first and second order derivatives M_i and M_{ij} ($i, j = 1, \ldots, n$) are given by

$$M_i = \frac{D_i d_i}{d} Y^{1-d} x_i^{-(1-d_i)} = D_i \frac{a}{a_i} \cdot \frac{1-a_i}{1-a} \cdot Y^{1/a} x_i^{-1/a_i}, \tag{A.2}$$

$$M_{ii} = \frac{M_i^2}{Y}\left(\frac{1}{a} - \frac{1}{k_i a_i}\right), \tag{A.3}$$

and

$$M_{ij} = \frac{M_i M_j}{Ya}, \tag{A.4}$$

where

$$i \neq j, \qquad a = \frac{1}{1-d}, \qquad a_i = \frac{1}{1-d_i}, \qquad k_i = \frac{M_i x_i}{Y} > 0.$$

Since positive marginal productivities M_i are required by the first order conditions, (A.2) implies $(D_i d_i/d) > 0$, all i; hence the sign of d must be equal to the sign of all $D_i d_i$ (which are all equal). In addition, for $M(\underline{x})$ to be non-negative and finite at all $\underline{x} \gg \underline{0}$, (A.1) implies $D_i > 0$, all i. This implies either $d_i < 0$, all i, and $d < 0$, *or* $d_i > 0$, all i, and $d > 0$. For the general case where some $D_j < 0$, there exist isoquants with $\underline{x} \geqslant \underline{0}$ corresponding to all finite, positive output levels, with the required monotonicity and convexity properties. However, the domain of $\underline{x}$ covered by these isoquants is restricted to a proper subset of the positive orthant of R_n. One could extend $M(\underline{x})$ to all $\underline{x} \geqslant \underline{0}$ in these cases, by defining $M(\underline{x}) = 0$ for all $\underline{0} \leqslant \underline{x} < \underline{x}'$ such that $M(\underline{x}') = 0$, and $M(\underline{x}) = \infty$ for all $\underline{x} > \underline{x}'' > \underline{0}$, such that $M(\underline{x}'') = \infty$.

The second order conditions for profit maximization (necessary and sufficient for interior regular maximum) require that the Hessian matrix $H_n = [M_{ij}]_n$ be negative definite, i.e., that

$$\text{(A.5)} \qquad (-1)^m |H_m| > 0 \qquad (m = 1, \ldots, n),$$

where $|H_m|$ is the m-order principal minor of H_n. From (A.3) and (A.4) we have

$$\text{(A.6)} \qquad H_n = [M_{ij}]_n = \left[\frac{M_i M_j}{Ya}\left(1 - \delta_{ij}\frac{a}{k_i a_i}\right)\right],$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Evaluation of $|H_m|$ yields, then,

$$\text{(A.7)} \qquad |H_m| = \frac{(-1)^m}{Y^m} \prod_{i=1}^{m} \left(\frac{M_i^2}{k_i}\right) \cdot \frac{a - \sum^m k_i a_i}{a \prod^m a_i} \qquad (m = 1, \ldots, n).$$

Thus the condition (A.5) implies here

$$\text{(A.8)} \qquad \frac{a - \sum^m k_i a_i}{a \prod^m a_i} > 0 \qquad (m = 1, \ldots, n).$$

Combining this with condition (17) in the text (and noting that $k_i = (M_i x_i/Y)$ is proportional to the share of costs $s_i = (p_i x_i/\Sigma\, p_k x_k) = (M_i x_i/\Sigma\, M_k x_k)$, we may now distinguish the following cases:

CASE (i): $a > 0$. By (A.8), for $m = 1$ we get $(a - k_i a_i)/a\, a_i > 0$, therefore $a_i > 0$, all i (i.e., a subcase of Case I).

In this case the necessary and sufficient condition for (strict) concavity is $\Sigma^m k_i a_i < a$, all m. Substituting $k_i = (M_i x_i/Y) = (p_i x_i/pY) \doteq (s_i \Sigma\, p_k x_k/pY)$ from the first order conditions for profit maximization, $pM_i = p_i$ (where $s_i = (p_i x_i/\Sigma\, p_k x_k)$ is the share in total costs and p is output price), we then have

$$\text{(A.9)} \qquad \left[\frac{\sum^n p_k x_k}{pY}\right] \sum^m s_i a_i < a \qquad (m = 1, \ldots, n).$$

Since $(\Sigma\, p_k x_k/pY)$ is at most equal to one (profits non-negative in long-run equilibrium), and $\Sigma\, s_i = 1$, we infer that if $a_i < a$, all i, condition (A.9) is satisfied. But if at least some $a_j > a$, then (A.9) implies restrictions on the domain of variation of the $\{x_i\}$, for any given set of parameters a, $\{a_i\}$. Outside this domain, concavity is not preserved.

CASE (ii): $a < 0$, all $a_i > 0$ (also a subcase of Case I). In this case $a - \Sigma^m k_i a_i < 0$, and $a\, \Pi^m a_i < 0$, all m, so that condition (A.8) is satisfied for all $(\underline{x})$.

CASE (iii): $a < 0$, a_1 (say) < 0, and $a_i > 0$ $(i = 2, \ldots, n)$ (corresponding to Case II). Factor x_1 is then a substitute to all others, whereas $x_2, \ldots, x_n$ constitute a group of complements, with $\sigma_{jk} < 0$; $j, k > 1$. In this case (A.8) implies $\Sigma_{i=1}^m k_i a_i < a < 0$, since $a\, \Pi_{i=1}^m a_i > 0$, or $k_1|a_1| > |a| + \Sigma_{j=2}^m k_j a_j > 0$. Substitution from (A.2) and rearrangements yield, finally,

$$\text{(A.10)} \qquad x_1 > \left[\sum_2^m \left|\frac{a_j D_j d_j}{a_1 D_1 d_1}\right| x_j^{d_j} + \left|\frac{a\, d\, Y^d}{D_1 d_1}\right|\right]^{1/d_1}.$$

which limits the domain of variation of x_1, for unit $h(Y)$, more than in equation (19), due to the additional positive term $|(a\, dY^d/D_1 d_1)|$ on the right.

Again, not more than one a_i may be negative, as can be seen from (17), or directly from (A.8). If $a_1 < 0, a_2 < 0, a < 0$, we have $(a - k_1 a_1)/aa_1 > 0$ and $(a - k_2 a_2)/aa_2 > 0$, which implies $[(a - k_1 a_1 - k_2 a_2)/(aa_1 a_2)] < 0$, contradicting (A.8) for $m = 2$. Thus the three cases given above cover all possible $(a, \{a_i\})$ combinations. The extensions to limiting cases with $a_i \to 0, 1$, or ∞ ($d_i \to \infty, 0$, or 1, respectively) are similar to the cases analyzed in the text. For the case $d = 1$ (or $a \to \infty$) we get

$$M_i = D_i d_i x_i^{-(1-d_i)}, \qquad M_{ii} = -\frac{D_i d_i}{a_i} x_i^{d_i - 2}, \qquad \text{and} \qquad M_{ij} = 0, \qquad i \neq j,$$

so that the second order conditions imply (since $M_i > 0, M_{ii} < 0$) $a_i > 0$, all i, and

$$(-1)^m |H_m| = (-1)^m \prod_{i=1}^{m} M_{ii} = \prod_{i=1}^{m} \frac{D_i d_i}{a_i} x_i^{(d_i - 2)} > 0, \qquad \text{all } m,$$

which is satisfied for all $\underline{x}$, as in the case (ii) above. (The cases $d \to 0$ or $-\infty$ are economically meaningless.)

The signs of individual d_i and D_i in the three cases are subject to the relations $D_i > 0$; and all d_i have the same sign as d.[26]

Returning to the conditions for cost minimization we have, using (A.2),

$$p_i = \lambda M_i = \frac{\lambda D_i d_i}{d} Y^{1-d} x_i^{-(1-d_i)} \qquad (i = 1, \ldots, n),$$

where $\lambda = \lambda(y; p_1, \ldots, p_n)$ is marginal cost. Solving for x_i (using the definitions of a_i, a) gives

(A.11) $\quad x_i = A_i p_i^{-a_i} Y^{a_i/a} \lambda^{a_i},$

where $A_i = [D_i d_i/d]^{a_i}$. For profit maximization $\lambda = p$, and we get[27]

$$x_i = A_i (p_i/p)^{-a_i} Y^{a_i/a}.$$

We may now evaluate the expansion elasticities of x_i by taking the partial logarithmic derivative of (A.11) (at constant factor prices $\{p_j\}$) with respect to Y:

(A.12) $$\frac{\partial \log x_i}{\partial \log Y} = a_i \left[\frac{1}{a} - \frac{\partial \log \lambda}{\partial \log Y} \right],$$

where $(\partial \log \lambda / \partial \log Y) > 0$ for profit maximization (concave $M(\underline{x})$).

Equation (A.12) indicates clearly the peculiarity of the function (1), as compared with the CRESH function. In CRESH (or any homothetic function), the elasticities of all inputs with respect to Y (at a given point) are *equal* (equal to one for CRS), since the expansion lines are straight lines through the origin. But for $M(\underline{x})$ these elasticities are different, and proportional to the parameters a_j. Since $(\sigma_{ij}/\sigma_{ik}) = (a_j/a_k) = (\eta_{x_j y}/\eta_{x_k y})$, we observe that the more factor x_j is a substitute to any given factor i, relative to factor x_k, the more the expansion line curvature in direction x_j, relative to direction x_k. In the special case $a_1 < 0$ (Case iii) under profit maximization we have $(\partial \log \lambda / \partial \log Y) > 0$, $1/a < 0$, so that the bracketed term in (A.12) is negative, and $(\partial \log x_1 / \partial \log Y) > 0$, but $(\partial \log x_j / \partial \log Y) < 0$ for $j = 2, \ldots, n$. Hence the group of complementary factors $\{x_2 \ldots x_n\}$ is also a group of *inferior* factors. With increasing output, their quantity decreases (factor prices constant), and with an increase in any of their prices p_j, marginal cost decreases, and the amount of output supplied perversely increases[28]—in the domain where the function $M(\underline{x})$ is concave.

[26] That is, if $M(\underline{x})$ is finite and positive for all $\underline{x} \gg \underline{0}$. In the more general case, the following weaker conditions are sufficient: $\max_i D_i > 0$, and all $D_i d_i$ have the same sign as d.

[27] These are equilibrium conditions, but not demand equations. If Y is solved from this system in terms of $\{p_i\}$, the resulting equations will be the factor demands $x_i(\underline{p}, p)$. Equations (A.12) which are linear in logs of p, p_i, and Y, could be used as a system of estimating equations. Alternatively, one could derive a set of $(n - 1)$ estimation equations under cost minimization, by elimination of λ between pairs of equations in (A.11). Cf. Cook [**4**], Dhrymes and Kurz [**6**], and Mukerji [**15**].

[28] See Footnote 15.

APPENDIX B: THE CRESH COST FUNCTION

We have argued above, that since the CRESH function is homothetic, its total cost function $c(\underline{p}, Y)$ is given by $c(\underline{p}, Y) = g(\underline{p}) \cdot h(Y)$, where $h(Y)$ is the same function appearing in definition (3). We now derive the unit cost function $g(\underline{p})$, which is linear homogeneous in $\underline{p}$, and independent of Y.

Define $\underline{x}(1)$ as the factor quantities on the "unit" isoquant $h(Y) = 1$ (where $y = \log h(Y) = 0$), and $\xi_i = \log x_i(1)$. $\underline{\xi}$ will now satisfy equations (3) and (8) for the case $y = 0$, i.e.,

(B.1) $$\sum D_i e^{d_i \xi_i} = 1,$$

(B.2) $$\xi_i = a_i(\Lambda - q_i + c_i).$$

Clearly, $g(\underline{p})$ is the cost function, where $h(Y) = 1$; hence

(B.3) $$g(\underline{p}) = \sum p_i x_i(1) = \sum p_i e^{\xi_i}.$$

Substituting (B.2) in (B.1) yields

(B.4) $$\sum D_i \exp\{d_i a_i(\Lambda - q_i + c_i)\} = 1,$$

and substituting (B.2) in (B.3) gives

(B.5) $$g(\underline{p}) = \sum p_i \exp\{a_i(\Lambda - q_i + c_i)\}.$$

Writing equations (B.4) and (B.5) in terms of original variables we finally get (since $d_i a_i = a_i - 1$)

(B.6) $$\sum^{n} A_i \left(\frac{\lambda}{p_i}\right)^{a_i - 1} = 1,$$

(B.7) $$g(\underline{p}) = \sum B_i p_i \left(\frac{\lambda}{p_i}\right)^{a_i},$$

where

$$A_i = D_i |D_i d_i|^{a_i - 1} \quad \text{and} \quad B_i = |D_i d_i|^{a_i} = |d_i A_i|.$$

Equation (B.6) now defines (implicitly) the function $\lambda(\underline{p})$ (where λ is the Lagrange multiplier of equation (4)), and equation (B.7) gives the unit cost function $g(\underline{p})$ in terms of $\underline{p}$ and $\lambda(\underline{p})$. Clearly, both $\lambda(\underline{p})$ and $g(\underline{p})$ are linear homogeneous in $\underline{p}$, and $(\partial g/\partial p_i) = x_i(1)$.

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