AN ECONOMETRIC INVESTIGATION OF THE TECHNOLOGY OF AGRICULTURAL PRODUCTION FUNCTIONS

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1. INTRODUCTION

Production theory long has held a central place in the literature of economics. Except for simple theories of single-commodity supply and demand functions, more has been written about either (1) the physical nature of (2) the implications of the algebraic form of the production function than about any other particular relationship of concern to economists. But while this large skeleton of theory exists, little empirical flesh has been fitted to it. Empirical production functions have been fitted to various types of cross-sectional and time-series observations, both for industrial and agricultural firms. Still, however, very little has been done to establish the physical nature of production coefficients. This is true even though the student in economics, from the undergraduate in his first course to the advanced graduate student, devotes an important part of his study to the implications of the physical production function.

To help fill these important but quite empty "shoeboxes," the writer has initiated and completed several production function studies in cooperation with physical scientists. These studies include controlled experiments with estimates by least-squares, single equation methods. The designs of the experiments have been based on mathematical models of the theory of production. The several reasons for initiating these studies include: (1) The data are useful for teaching purposes, at both the undergraduate and graduate levels. In providing some notion of the empirical nature of physical production functions, reality is attached to the purely logical statements which have claimed so much of the student's time. (2) The results provide fundamental knowledge of the physical world. (3) A saving in public funds results. Physical scientists have concentrated on providing point estimates which are most practical; and they have been efficient in doing so. At the same time, however, they have been carrying on experiments relating to the phenomena concerned but have employed models which suppose the observations to be discrete. Their general approach, in which a large proportion of Land Grant College research resources have been invested, generally provides a notion of a very few points on the production surfaces; and ordinarily to not lend themselves to economic interpretation by farmers who must use them. Since society already is investing experimental funds for these purposes, more information can be obtained from the same public resources by adopting procedures and estimates of the type outlined in this paper. (4) The data are basic for decisions by managers of farm firms.

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2. EXPERIMENTS COMPLETED AND UNDERWAY

This paper is an attempt to summarize and interpret briefly the econometric results from the several studies initiated to date. Because of space limitations, no attempt will be made to outline the details of experimental designs or to present all of the equations employed before particular algebraic forms were selected as providing the most efficient estimates for each set of observations. The extreme details of experimental design and statistical analysis can be found elsewhere.[2] To date, 14 experiments have been completed for fertilization of crops and four new ones are being initiated. These experiments include as many as four and as few as two variable nutrients. Two experiments have been completed for hogs, one in drylot and one on pasture; with the two major categories of feeds, corn and fortified protein feed (soybean oilmeal), as variables. An additional experiment has been initiated to deal with other feed alternatives. One experiment each has been conducted for broilers (young chickens) and turkeys. Again, these two studies include grain and protein feed as the variable resources around which main economic decisions revolve. One experiment has been completed with dairy cows and a new one has been initiated. These studies include concentrate (grain) feeds, forage feed, the inherent capacity of cows, and time in lactation period as variables. Several sets of data are being analyzed and a new experiment is being initiated for feeder (beef) cattle; with hay, corn, and protein feed as independent variables. In all of these studies, the factors mentioned above are considered to be those which farmers can control. Observations for these variables are controlled in the experiments; their only variance is that incorporated into the experimental design. The attempt is to predict yield per acre for crops, gain per bird or animal for chickens, turkeys, hogs and beef cattle, and milk production per cow for dairy animals.

In each experiment, an algebraic form of function was selected which was considered within the logic of plant and animal response and yet appeared to give the most efficient estimates. Several algebraic forms of equations were tried for each experiment. It is not claimed, however, that the mathematical forms presented provide the final biological representation of the phenomena concerned. Numerous statistical criteria were used in selecting the production functions used for prediction. In most of the studies, alternative algebraic forms which were computed have been provided for persons who wish to use estimates based on other functions. After selection of an algebraic form of production

[2] See the following: Heady, Earl O., John T. Pesek, and W. G. Brown, "Crop Response Surfaces and Economic Optima in Fertilizer Use," Iowa Agr. Exp. Sta. Bul. 424. Heady, Earl O., *et al.*, "Feed Substitute Rates and Economic Efficiency in Pork Production," Iowa Agr. Exp. Sta. Bul. 409. Brown, W. G., Earl O. Heady, and John T. Pesek, "Production Functions, Isoquants and Isoclines in Fertilizer Use," Iowa Agr. Exp. Sta. Bul. 439. Heady, Earl O., *et al.*, "Production Functions, Gain Isoquants and Least-Cost Rations for Broilers," Iowa Agr. Exp. Sta. Bul. 442. Heady, Earl O, John Schnittker, and Norman L. Jacobson, "Milk Production Functions, Hay/Grain Substitution Rates and Economic Optima in Dairy Cow Rations," Iowa Agr. Exp. Sta. Bul. 444. Heady, Earl O., *et al.*, "Production Functions, Gain Isoquants and Least-Cost Ration for Turkeys," Iowa Agr. Exp. Sta. Bul. 443.

function, production surfaces, isoquants, isoclines, ridge lines, convergence points, and other quantities of physical and economic interest have been computed to serve as a basis for economic decisions. Popular presentations, allowing farmers to equate derivatives or marginal quantities with price ratios, have been worked out and are beginning to receive wide acceptance.

3. RESULTS OF EXPERIMENTS

Each type of physical phenomenon, or the practical uses to be made of it, requires particular characteristics in the form of the production function. For example, the fertilizer production surface has (a) definite ridge lines (isoclines denoting a zero marginal rate of substitution between factors), (b) marginal rate of substitution which change along a scale line, and (c) a point of maximum yield per acre where the derivatives of all factors are simultaneously zero.[3] It requires an algebraic form of function which allows the production surface to form a peak. A power function of the Cobb-Douglas type, or an exponential equation, does not allow this combination of conditions. Slopes of isoquants for a power function are constant along a scale line; the surface forms a ridge, approaching the limit of total production, rather than a peak; the restrictions of constant elasticity or constant marginal product ratios are inconsistent with the underlying biological logic. Milk production parallels somewhat the surface of crop fertilization. Hence, a conventional quadratic equation, or some transformation of it may best describe the production surface. In contrast, however, the production surface, expressing output of meat per bird or animal as a function of feed inputs, logically should approach a ridge, rather than converge to a single peak. Still, however, even for meat production, it is unlikely that the isoclines are linear through the origin, a restriction imposed by a power function. Under linear isoclines through the origin of the feed plane, the ratio of feeds (ration) which allows least-cost gains for small birds or animals also would allow least-cost gains for those approaching maturity. Past investigations show, however, that for a small growing animal, protein requirements are high relative to carbohydrate feeds; as the fattening or finishing stage is approached and attained, requirements for gains shift in the direction of carbohydrate feeds. With this obvious change in the substitution rate between major classes of feeds, a function is required which allows nonlinear isoclines; or at least linear isoclines which do not pass through the origin of the feed plane. However, as is pointed out later, practical uses of data in decisions may allow functions which provide linear isoclines over intervals of the production surface. In fitting functions to the various sets of technical phenomena, it was found that the elasticity of production is generally less than 1.0 over all ranges of inputs for particular resources and, therefore, terms need not be included in the regression equations to allow for increasing

[3] We use the term scale line for purposes of abbreviation in this paper. We do not refer to the conventional use where there are no fixed factors and all factors are increased by the same proportion, but instead to the case where the variables under consideration are increased in fixed proportions; but the animal, bird, or acre is held in fixed amount for the experiment. This modified meaning is used in the remainder of this paper.

marginal productivity. Results for various functions, with some interpretation of their biological and economic nature, follow.

Experiments with Fertilization of Crops

One of the most successful experiments in fertilization of crops was conducted on corn on calcareous Ida silt loam in western Iowa. In 1953, nine rates each of elemental nitrogen (N) and phosphate (P) were applied to corn. The complete block design originally allowed for 81 combinations of the nutrients, although an incomplete block design was finally used with 63 combinations. Yield measurements were obtained in 1953, the year of application, and also in 1954 when additional applications were not made. The regression equation providing most efficient predictions for the first year was that shown in (1)

$$(1)\quad \underset{}{Y = -5.682} - \underset{(7.9)}{.316N} - \underset{(10.4)}{.417P} + \underset{(7.3)}{6.3512\sqrt{N}} + \underset{(9.8)}{8.5155\sqrt{P}} + \underset{(8.9)}{.3410\sqrt{PN}}$$

where Y refers to corn yield per acre in bushels and N and P refer respectively to the pounds of elemental nitrogen and phosphate applied per acre. The t values (shown in parentheses below the regression coefficients) were all significant at probability levels less than .01 while the value of R was .9582. The precision of the estimates is very great, considering the biological nature of the crop production process and variations which exist because of soil heterogeneity, insects, weather, and similar variables. The partial derivatives in (2) and (3),

$$(2)\qquad \partial Y/\partial N = -.316 + 3.1756N^{-.5} + .1705N^{-.5}P^{.5},$$

$$(3)\qquad \partial Y/\partial N = -.417 + 4.2578P^{-.5} + .1705N^{.5}P^{-.5},$$

have been used in simultaneously specifying the optimum level of fertilization and the optimum combination of fertilizer nutrients. The latter show (a) the marginal product of N to become zero at 101.0 pounds when this nutrient alone is used, (b) the marginal product of P to become zero at 104.3 pounds when it alone is used, and (c) that the marginal products of the two nutrients do not simultaneously drop to zero until 397.6 pounds of N and 336.6 pounds of P are used in combination. However, equation of the partial derivatives with the respective resource to product price ratios shows the optimum inputs of N and P to be considerably less than the latter quantities. Equations (4) and (5),

$$(4)\quad N = [10.05 + .539\sqrt{P} \pm \sqrt{-.4115P + 15.0996\sqrt{P} - 1.2645Y + 33.153}]^2,$$

$$(5)\qquad \frac{\partial N}{\partial P}\ \frac{-.417 + 4.2576P^{-.5} + .1705N^{.5}P^{-.5}}{-.316 + 3.1756N^{-.5} + .1705N^{-.5}P^{.5}},$$

have been derived to provide estimates respectively of the isoquants and isoclines (with a modification of the latter used for predicting marginal rates of nutrient substitution along an isoquant of given yield level). Figures 1 and 2 have been derived respectively from equations (4) and (5). These figures, and the magnitude of the coefficients in their underlying equations, indicate particular

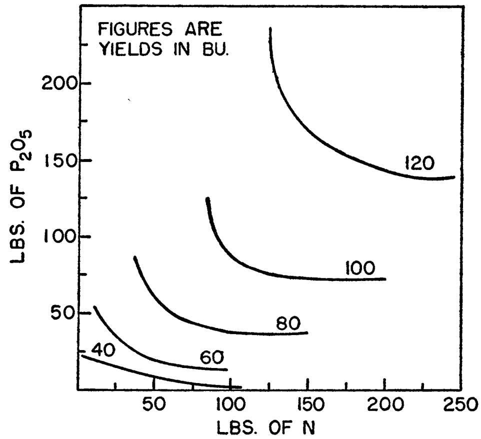

Figure 1.—Corn Isoquants with N and P_2O_5 Variable from Equation (1)

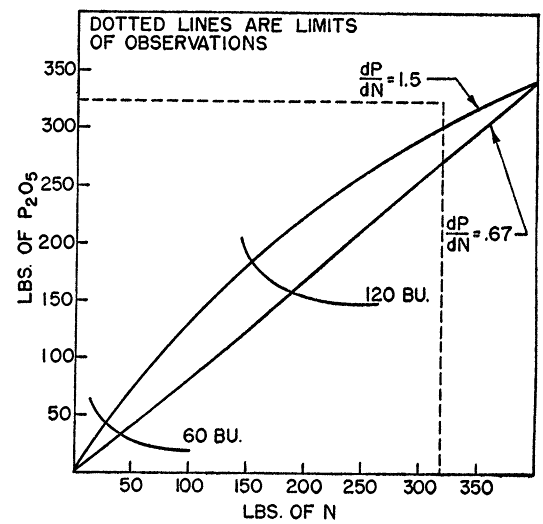

Figure 2.—Corn Isoclines from Equation (1). Derivatives Indicate Marginal Rate of Substitution for Two Isoclines.

characteristics of the biological phenomena and the economic recommendations which might be based on them. As Figure 2 shows, the isoclines are generally curved, converging at the point of maximum yield per acre. The ridge lines (i.e., isoclines of $dN/dP = 0$ and $dP/dN = 0$) fall approximately at the end of the isoquants shown in Figure 1. The two isoclines (i.e., expansion paths or lines denoting equal substitution rates) are the extremes of all N/P price ratios which might be reasonably expected; the upper and lower ones representing respectively dN/dP values of 1.5 and .67. The isoclines neither curve sharply nor bend far apart between the magnitudes shown. Neither do the isoquants change greatly in slope between the two isoclines (i.e., changes in marginal rates of substitution are relatively small between these limits). Hence, it is true that (a) while least-cost nutrient combinations do differ with price ratios, profit sacrifices for deviations from the equality dN/dP = price $P \div$ price N are not great for a particular yield, and (b) while slight changes should be made if the least-cost factor combination is to be used as yields progress to higher levels, profit sacrifice is not great if the same nutrient ratio (factor combination) is used for alternative levels of yield. To find that numerous combinations exist, even though some of them represent deviations from economic optima, which give insignificant differences in profit per acre is of value. As a matter of practicality, agriculturists often suggest a single nutrient combination, regardless of the level of yield to be attained, since farmers have varying amounts of capital and many cannot push fertilization to a level where the ratios of added product per unit of factor are equated to price ratios. Hence, knowledge that isoquant and isocline configurations allow considerable swings in nutrient price ratios or yield levels without causing more than a few cents deviation from economic optima, if a single nutrient ratio is used, is itself positive and useful knowledge. (While these conditions do in fact exist for the data explained above, experiments for other years, crops, and location show that the mathematical nature of other production functions may call for considerable differences in the nutrient combination as price ratios and capital availability change.)

The partial derivatives in (2) and (3) were equated to the nitrogen to corn and the potash to corn price ratios existing at the time of the experiment. (Prices were \$1.40 for corn, \$0.18 for N, and \$0.12 for P.) Solving for N and P, it is shown that optimum fertilization includes 142.5 pounds of N and 156.5 pounds of P, with a predicted yield of 117.2 bushels of corn. With the prices of N and P reversed, optimum fertilization includes 208.7 pounds of N and 175.5 pounds of P and a yield of 124.9 bushels. With the first mentioned prices for N and P, optimum corn yield is 105 and 124.9 bushels respectively when corn is priced at \$0.90 and \$2.00 per bushel.

A fertilizer-crop production function which gives a quite different isocline map is that of (6),

$$(6)\quad Y = 1.874 - \underset{(2.0)}{.0014K} - \underset{(6.8)}{.0050P} + \underset{(3.9)}{.0617\sqrt{K}} + \underset{(10.8)}{.1735\sqrt{P}} - \underset{(2.3)}{.00144\sqrt{KP}},$$

for alfalfa hay where Y is hay yield in ton per acre, K is potash in pounds per acre, and P is phosphate in pounds per acre. The experiment upon which these predictions are based was conducted in 1953 in north Iowa and included an experimental layout like that explained above for corn. All t values (in parentheses) are significant at probability levels of less than .05, and the value of R is .8793. As for corn, the best fitting algebraic form was a quadratic equation with square root transformations. This function gives rise to the isoquants and isoclines shown in Figure 3. When isoclines have sharp curvature such as these, and are "sprung apart widely" over ranges of relevant price ratios for nutrients, considerable changes may need to be made (or profit sacrifices may be large in failure) to adjust nutrient ratios as their price ratio changes, or as alternative levels of

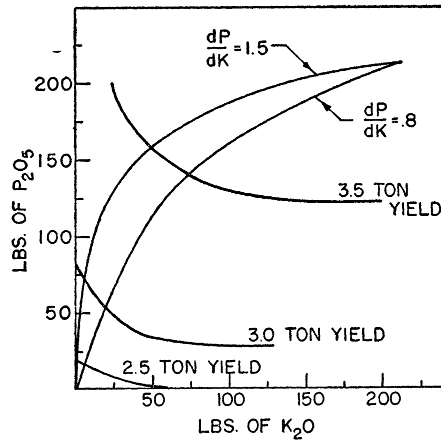

FIGURE 3.—Isoquants and Isoclines for Alfalfa from Equation (6)

yield become profitable through changes in the product price. As the isoclines and isoquants indicate, the soil was more deficient in P than K (i.e., smaller amounts of the former were already available in the soil). As yield level is pushed upward, however, K becomes more nearly a limiting resource and the input of it must be increased sharply, if a given yield is to be attained with minimum cost. Setting the partial derivatives of (6) to equal the product to resource price ratios existing at the time of the experiment in (7) and (8),

$$(7) \qquad \partial Y/\partial K = -.00139 + .03085K^{-.5} - .00072K^{-.5}P^{5} = .15/16.00,$$

$$(8) \qquad \partial Y/\partial P = 0.00502 + .08676P^{-.5} - .00072K^{.5}P^{-.5} = .12/16.00,$$

the optimum quantities of P and K are predicted to be respectively 63.4 pounds and 8 pounds, with a yield of 3.07 tons. For yields this low, however, the profit sacrifice is not great if only P is used. Of course, the functions can be used to predict economic optima for other price situations.

A fertilizer production function which gives quite different relationships is (9),

$$(9) \quad Y = 60.83 + \underset{(5.7)}{.7126N} - \underset{(2.9)}{.00435N^2} + \underset{(2.0)}{.5255P} - \underset{(1.9)}{.003103P^2} + .2546K - \underset{(2.1)}{.001624K^2} - \underset{(2.1)}{.002255PK},$$

for corn on Haynie soil in 1954 with nitrogen, phosphate, and potash as variable nutrients and stand fixed at 16,000 plants per acre. The t values for this $3 \times 3 \times 3$ randomized block design with replication and 54 observations were all significant at a probability level of .04 or less. While the interaction terms for N with the nutrients were not significant, and consequently were dropped from the regression equations, the P-K interaction term was significant. Production surfaces for P and K, with nitrogen fixed at three levels, are shown in Figures 4, 5, and 6. Isoquants and isoclines for N and K as variables, with P fixed at zero, are

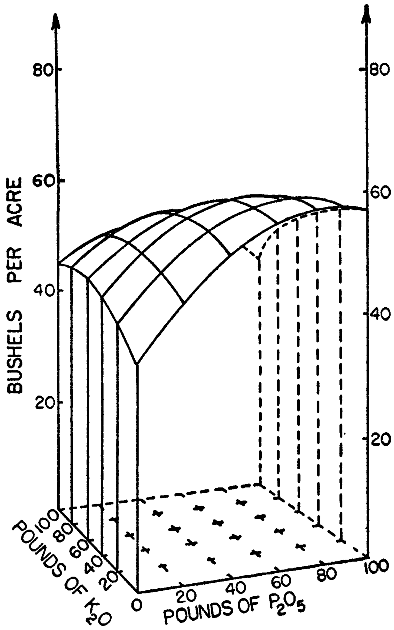

Fig. 4

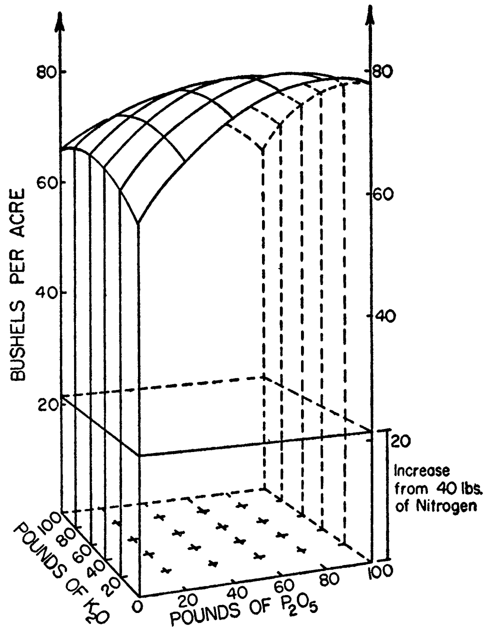

Fig. 5

FIGURE 4.—Predicted *P-K* Surface for Equation (9) with N at Zero

FIGURE 5.—Predicted *P-K* Surface for Equation (9) with N at 40 Lbs

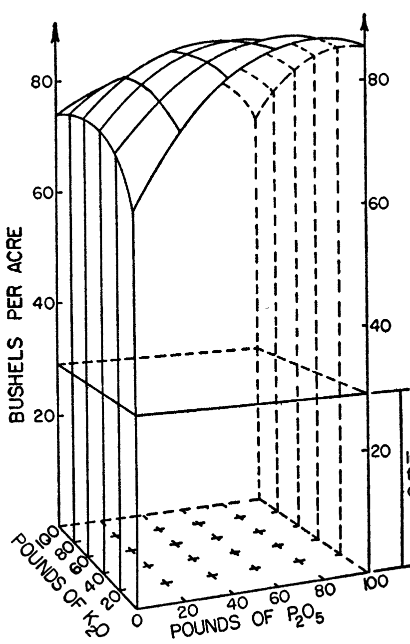

FIGURE 6.—Predicted *P-K* Surface for Equation (9) with N at 80 Lbs

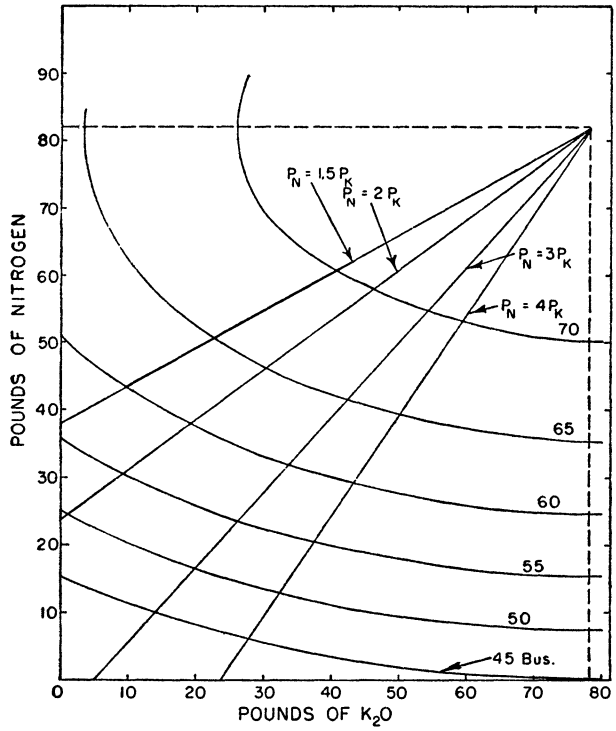

FIGURE 7.—Yield Isoclines and Isoquants for Equation (9). N and K_2O Variable with P_2O_5 at Zero. (Dashed Lines are Ridge Lines.)

shown in Figure 7. The slope of the isoquants changes materially for successively higher yield levels, with N as well as K becoming limitational resources on higher isoquants. Since no N-K interaction terms are included in the particular form of equation, the isoclines denoting $dP/dN = 0$ and $dN/dP = 0$ (i.e., ridge lines) are horizontal and vertical respectively. Isoquants and isoclines for N and P, with K fixed at zero, are similar to those in Figure 7; while those for P and K, with N fixed at different levels, differ somewhat in general configuration. The conventional quadratic equation of (9) results in linear isoclines; in contrast to the square root transformations in (1) and (6) where they are non-linear. This is partly obvious in the derivatives of (9) provided in (10), (11), and (12),

(10) $$\frac{\partial P}{\partial N} = -\frac{.7126 - .00870N}{.5255 - .00621P - .00226K},$$

(11) $$\frac{\partial K}{\partial N} = -\frac{.7126 - .00870N}{.2546 - .00325K - .00226P},$$

(12) $$\frac{\partial P}{\partial K} = -\frac{.2546 - .00325K - .00226P}{.5255 - .00621P - .00226K}.$$

These equations, which define the marginal rates of substitution between the three pairs of the nutrients have only linear terms for the nutrients included. Linearity of expansion paths is even more evident in (13) and (14)

$$(13) \qquad P = \frac{(.525\propto - .713)}{006.\propto} + \frac{.009}{.006\propto} N - .364K$$

$$(14) \qquad N = (81.87 - 60.37\propto) + (.713\propto)P + (.260\propto)K,$$

which are isocline equations, computed by setting the derivatives in (10), (11), and (12) to equal a constant substitution ratio of $\propto$, and expressing the amount of one nutrient as a function of the other (i.e., predicting all nutrient combinations which give the specified marginal rate of substitution or price ratio). Since (13) and (14) have only linear terms for K and P, the isoclines are linear, intersecting the nutrient axes at the levels specified by the constant first terms in the equations.

From the several experiments, and from comparison of functions which provide linear and curved isoclines converging at the point of maximum yield per acre, this tentative conclusion appears to hold true: If nutrient levels (and the yield without fertilizer) are very low, the best fitting function will be the one with square root transformations and curved isoclines. This conclusion appears logical since, starting from zero inputs of either nutrient and a zero yield (i.e., isoclines which must pass through the origin and cannot "fan out" but must converge to a point of maximum yield), the isocline family would otherwise reduce to a single straight line, a condition which biologically is illogical. However, where yields without added fertilizer nutrients are quite high, and increments can be made to yield by addition of either nutrient alone within reasonable production levels, the best estimating equation is one with isoclines which are linear and intersect the axes. While there may be very slight curvature in these isoclines, the curvature is insufficient to cause square root terms to be more efficient than squared terms as estimators in the overall production function.

Experiment in Milk Production

Milk production parallels somewhat the technological conditions of crops. Apparently, there are ridge lines with (1) $dG/dH = 0$, where G refers to grain consumed and H refers to forage consumed per cow, defined by the limit of the cow's stomach capacity to consume the more bulky hay and still allow attainment of a given milk yield, and (2) $dH/dG = 0$ defined by the physiological minimum of forage which is necessary for health of a ruminant animal. Within the limit of these ridge lines, the isoclines must converge to a single point where (a) maximum milk production per cow is attained and (b) the two classes of feeds are both limitational (i.e., are technical complements where the maximum output isoquant reduces to a single point). With these conditions in mind, appropriate algebraic equations were fitted to observations from 36 Holstein dairy cows on an experiment where the 36 treatments included cows of three levels of inherent ability, 4 rations (ratios of concentrates and forage) with three levels of feeding each. The production function selected as most appropriate for predictions is included in (15),

$$(15)\quad M = \underset{(1.94)}{1.6302H} + \underset{(2.60)}{3.1309G} + \underset{(9.86)}{.1479A} - \underset{(.34)}{14.2243T} - \underset{(1.14)}{.000388H^2} - \underset{(1.67)}{.001192G^2}$$

$$- \underset{(1.62)}{4.3792T^2} - \underset{(1.10)}{.001056HG} - \underset{(2.82)}{.1570GT} - \underset{(3.49)}{.0865HT} - 731.76,$$

where M refers to milk production per month per cow, H refers to the pounds of forage (in the form of legume hay), C refers to the pounds of concentrate (grain) mix consumed per cow, A refers to the ability of cows measured as output per cow in a preliminary production period of one month when all cows were on the same ration, and T refers to time in terms of months during the six-month experimental period. The value of R for these variables was .9016. While the standard errors were relatively high (the figures in parentheses are t values) for some of the variables included in the predicting equation, they were left in the equation. This is because they gave estimates corresponding more nearly to biological logic and previous nutrition knowledge than (a) modified equations which did not include these terms or (b) alternative equations where all standard errors were relatively low and all t values were significant at probability levels of .05 or .01.

From equation (15) it is possible to predict (a) least-cost feed combinations for

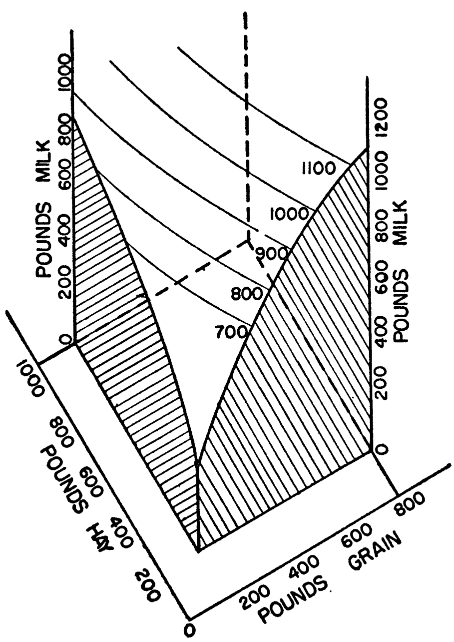

FIGURE 8.—Milk Production Surface from Equation (15). Ability and Time Set at Mean.

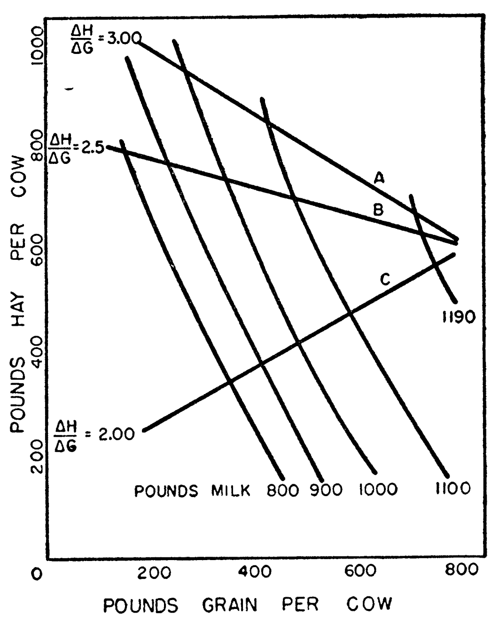

FIGURE 9.—Isoquants and Isoclines for Mean Month of Experiment from Equation (15). Isoclines for Substitution Rates Indicated.

particular levels of milk output and (b) the most profitable level of feeding for any point in time of the lactation period for cows of various levels of inherent ability. Figure 8 includes a production surface with milk as a function of hay and concentrate consumption for the midpoint in time of the experiment ($T = 3.5$) with cow production ability fixed at the mean of the group of animals. A parallel map of milk isoquants and isoclines or expansion paths is included in Figure 9. Maximum milk production in the "mean month" of the experiment approximates 1200 pounds per cow with a ration of 600 pounds of hay and 750 pounds of grain fed throughout the month. An indication of how marginal rates of substitution differ between milk isoquants and between months of the lactation period are given in Tables I and II. The feed combinations shown provide the basis for the milk isoquants and are derived from

$$(16)\quad H = 1989.36 - 1.3608C \pm (-1288.66)\sqrt{1.8553 + .0014G - .0000007G^2 - .0016M}.$$

The marginal rates of substitution (derivatives) are for "exactly" the feed combinations indicated and are from

$$(17)\qquad \frac{\partial H}{\partial G} = \frac{2.9740 - .0024G - .0011H}{1.5437 - .00078H - .00106G}.$$

Within any month of the experiment, feed combinations included for each milk isoquant are limited to those which are consistent with biological possibilities. For higher levels of milk production, the grain minimum is lower and the hay maximum is higher; the limited capacity of the cow's stomach would not allow

TABLE I

FEED COMBINATIONS AND MARGINAL RATES OF SUBSTITUTION FOR SPECIFIED MILK ISOQUANTS AND FIRST MONTH IN TIME (T-1). COW ABILITY AT MEAN OF GROUP

Level of grain (lbs.)	Lbs. of hay, with grain listed at left, for milk isoquant of:			Marginal rates of substitution dH/dG for feed combinations shown and milk isoquant of:		
	1000 lbs.	1200 lbs.	1400 lbs.	1000 lbs.	1200 lbs.	1400 lbs.
150	883			2.41		
200	765	1093		2.29	2.77	
250	654	960		2.18	2.55	
300	547	837		2.09	2.38	
350	445	722		2.01	2.24	
400	346	613	1024	1.94	2.13	2.87
450	250	509	889	1.87	2.02	2.54
500	158	410	768	1.81	1.93	2.31
550		315	657		1.85	2.14
600		225	554		1.78	2.00
650			457			1.87
700			366			1.76
750			280			1.67
800			199			1.57

TABLE II

FEED COMBINATIONS AND MARGINAL RATES OF SUBSTITUTION FOR SPECIFIED MILK ISOQUANTS AND SIXTH MONTH IN TIME (T-6). COW ABILITY AT MEAN OF GROUP

Level of grain (lbs.)	Lbs. of hay with grain listed at left, for milk isoquant of:			Marginal rates of substitution dH/dG for feed combinations shown and milk isoquants of:		
	700 lbs.	800 lbs.	900 lbs.	700 lbs.	800 lbs.	900 lbs.
150	801			2.98		
200	661			2.62		
250	537	867		2.38	3.88	
300	422	701		2.20	2.93	
350	316	566		2.06	2.50	
400	216	448		1.93	2.23	
450		341			2.04	
500		244	732		1.88	14.76
550			530			2.67
600			414			2.06
650			320			1.71
700			242			1.44
750			176			1.18

the higher levels of milk production to be produced with "bulky" hay inputs. As is expected the marginal rates of substitution of grain for hay are generally greatest for higher milk isoquants, as well as for rations denoting a low proportion of grain to hay along a given isoquant. Between months of the lactation period, the marginal rate of substitution of grain for hay increases, although the marginal productivities of both feeds decline with time as the cow approaches the end of her "annual production period." (The isoquant levels are lower for the 6th than for the 1st month because of the same reason: The 1st month levels cannot be attained in the 6th month.) Dairy specialists are impressed by the slow rate at which the marginal rates of substitution change (the relatively small degree of curvature in the isoquants) for the feed combinations which are relevant to each milk level. Given usual prices for hay and grain and the cost of labor for controlled feeding of hay, the least-cost ration may often be attained, because of the isoquant slopes, by selecting an economic optimum grain quantity, and letting the cow eat hay to the limits of her stomach capacity. However, as an aid to farmers in their decision making, optimum rations have been calculated for numerous factor-factor and factor-product price ratios.

Experiment in Hog Production

Results for only one of the several experiments in hog feeding will be reported here. This experiment included 302 hogs and was conducted with corn, a feed high in carbohydrate content, and soybean oilmeal, a feed high in protein content, fed in drylot. Hogs were carried on the experiment from the weaning age of 34 pounds to a marketing weight of 200 pounds.

Certain modifications of the approaches outlined above were used for hogs. The main problem in hog production is to find the least-cost ration for different weights and to feed the hogs to the final weight which gives the highest market grade and prices. In other words, hogs reach a more or less distinct market weight, with a substantially lower price for hogs either lighter or heavier. The time required for attainment of a given weight also depends on the ratio of carbohydrates and protein fed. Thus, the optimum total feed inputs per hog cannot be solved simply by equating (a) the partial derivatives of animal gain in respect to feeds with (b) the feed to pork price ratios. Then, too, the problem is different from the fertilization phenomena in the following respect. For crops, the farmer can determine the optimum fertilization program, apply this amount and, aside from random fluctuations due to weather, etc., harvest a particular level of crop. However, for a meat animal, feed representing the entire path of inputs and outputs must be poured into the animal. (In contrast, for fertilization the farmer does not apply a small amount of fertilizer and obtain a small yield, then add a little more fertilizer to carry yield a little higher and repeat the process until the optimum yield is obtained.)

For crop production the least-cost combination of nutrients can be predicted for the "final" yield level; but for hogs, a least-cost ration should be determined for each weight which the hog will attain in progressing to market weight. However, as a practical matter, and in consideration of labor costs, the farmer ordinarily changes the ration about three times. He decides on a least-cost ration for small weights, feeds this ration over a range of gains, then repeats the process for a couple of other weight ranges, up to market weight. Obviously, this process does not give the least-cost use of feeds for each ounce of gain. In practical terms, however, a ration which averages least-cost over a weight range is needed. This consideration was included in the fitting and selection of functions. Several quadratic types of equations were fitted which appeared statistically efficient. However, all of these had either linear isoclines which did not intersect the origin of the feed plane, or nonlinear isoclines which did intersect the origin. Physiologically, these conditions relating to the isoclines are desirable, since they specify that the marginal rates of substitution change as the hog is taken to higher weights. This biological condition holds true since at lower weights the pig is in the growing stage and needs protein relative to carbohydrates; but near marketing age, the hog is in the fattening stage and requires carbohydrates relative to protein. Yet quadratic-based isoclines which give constant changes in feed ratios do not correspond to the practical considerations outlined above. For example, the farmer may wish to feed a single ration (proportion of feeds) over the weight range 50–100 pounds. However, selection of the optimum ration at the point of intersection of an isocline (based on a quadratic equation) with the 100 pound weight isoquant would not give the single ration which averages least in cost over the entire 50–100 pound weight range. A power function would allow selection of such an "average least-cost ration" for all pounds in the weight range (although it need not give the least-cost ration for any particular pound in the range). This is true since its isoclines are linear and pass through the origin of the feed plane. The difficulty of a single power function fitted to the entire set of observations revolves around the characteristics of the isoclines, however. They specify feeding a single ration over the entire production period, regardless of changes in nutritional requirements between growing and fattening stages, and consequently in the marginal rates of substitution between feeds. Hence, an alternative was employed to meet the conditions and requirements set out above. This was to fit a power function to the observations of three weight ranges in the growth-to-fattening period. This procedure allowed selection of an "average" least-cost ration to be specified within a weight range, but allowed selection, by equating derivatives along an isoquant with feed price ratios, of a different ration for heavier weight intervals. In effect, this procedure breaks the isocline into three linear segments, much in the nature of linear programming concepts. The resulting three "interval" power production functions are those listed in (18), (19), and (20).

(18) 34–75 lb. weight interval: $$G = 1.605 \underset{(14.3)}{P^{.297}} \underset{(19.1)}{C^{.533}},$$

(19) 75–150 lb. weight interval: $$G = .714 \underset{(8.7)}{P^{.142}} \underset{(31.4)}{C^{.767}},$$

(20) 150–250 lb. weight interval: $$G = .459 \underset{(5.3)}{P^{.092}} \underset{(26.9)}{C^{.856}}.$$

In these equations, G refers to gain in pounds per pig within the weight interval, P refers to pounds of soybean oilmeal, and C refers to pounds of corn consumed per pig within the weight interval. As indicated by the t values (included in parentheses below the relevant coefficient), regressions were all significant at probability levels of less than .01.

Some interesting technological phenomena are apparent in the coefficients: (1) The elasticities for protein (soybean oilmeal) decline over the weight intervals, in line with the nutritional considerations outlined above. (2) The elasticities for corn increase over the weight ranges, corresponding to the need for carbohydrates in the finishing process. These differences are illustrated more vividly in the corresponding isoquant equations of (21), (22), and (23), respectively,

(21) $$C = \left(\frac{G}{1.605P^{.297}}\right)^{1.876},$$

(22) $$C = \left(\frac{G}{.714P^{.142}}\right)^{1.213},$$

(23) $$C = \left(\frac{G}{.459P^{.092}}\right)^{1.167}$$

and the corresponding marginal rate of substitution equations derived from them in (24), (25), and (26), respectively:

(24) $$\partial C/\partial P = .557\ C/P,$$

(25) $$\partial C/\partial P = .185\ C/P,$$

(26) $$\partial C/\partial P = .108\ C/P,$$

Again, the magnitudes of these equations change consistently over the weight intervals and, as the latter set of equations shows, the marginal rate of substitution of soybean oilmeal for corn declines as the hog becomes heavier in weight.

While there are limitations in the use of a single power function (over all ranges of observations) for estimating marginal production for a given ration (mainly because the equation overestimates the surface for large feed inputs) this difficulty is lessened in (a) use of the interval functions which allows each segment of the function to fit closely with the observations or (b) use of another function which does not encounter these difficulties. Both methods were used and added precision to predictions. While numerous tabular data have been derived for the use of farmers and feed manufacturers, an "econometric device" has been worked out and put into production for farmers. A picture of it is shown in Figure 10. It includes data on both sides of a disc for determining the least-cost ration in the different weight intervals and in predicting optimum market weights. In effect, it allows the farmer to equate the derivative of corn

FIGURE 10.—Pork Costulator for Determining Least-Cost Rations

with respect to soybean oilmeal with the soybean oilmeal to corn price ratio. This is accomplished by turning the disc until the current prices of corn and protein supplement are in mesh at the edge of the wheel. Within the slots are data showing the amount of feed which meet the mathematical conditions for optima. The data also indicate the amount of time required for marketing under each ration, since the rate of growth varies with rations. If the farmer expects a price break, he may wish to feed protein in excess of that for least-cost gains, in order to take advantage of higher prices before the market break. Time functions were computed as the basis for these predictions but are not included here because of space limitations.

Experiment with Broilers

The broiler production problem is similar to that for hogs. Producers usually change the ration twice over the entire production period. Hence, they wish to find a ration which "averages least-cost" over a weight range, rather than to minimize cost of feed for each successive ounce of gain. Equations

$$\text{(27)} \quad \text{Overall:} \quad G = .033 + \underset{(39.8)}{.4823C} + \underset{(26.7)}{.6415P} - \underset{(7.4)}{.0183C^2} - \underset{(7.44)}{.0497P^2} - \underset{(3.28)}{.0232CS},$$

$$\text{(28)} \quad \text{Overall:} \quad G = .992\underset{(26.8)}{P^{.337}}\underset{(43.7)}{C^{.554}},$$

$$\text{(29)} \quad \text{Interval up to 1.3 lbs.:} \quad G = 1.075\underset{(46.5)}{P^{.384}}\underset{(33.1)}{C^{.543}},$$

$$\text{(30)} \quad \text{Interval over 1.3 lbs.:} \quad G = .702\underset{(18.0)}{P^{.294}}\underset{(38.1)}{C^{.646}},$$

include four production functions derived from an experiment with 600 chickens where the feeds or symbols have the same meaning as for hogs. (Somewhat different vitamins and mineral supplements were used in fortifying the soybean oilmeal, the main source of protein, and gain quantities are on a per bird basis.) The first and second equations below are overall functions fitted to the observations over the entire growth period, while the third and fourth equations are interval functions similar to those for hogs. Generally the t values (in parentheses) are significant at probability levels of less than .01. All values of R were large, with those for equations (29) and (30) being .9956 and .9885 respectively.

Another interesting illustration of the effect of nutritional requirements on the marginal productivity of feeds is given in Figure 11. Rations high in protein contain a large proportion of soybean oilmeal relative to corn; the oilmeal contains about 42 per cent protein while the corn contains only around 8 per cent. A ration of given percentage protein represents a fixed ratio of the two feed resources, soybean oilmeal and corn. As the marginal productivity lines again indicate, rations (a fixed ratio of two feeds) high in protein have a higher marginal productivity than those low in protein for small feed inputs (and hence over low bird weights). However, as the growing period progresses to maturity,

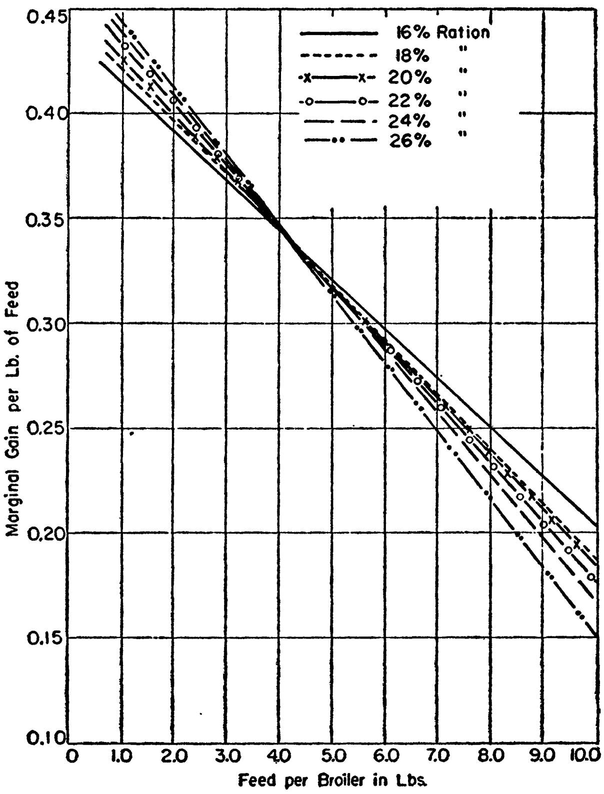

FIGURE 11.—Marginal Gains Per Pound for Broiler on Various Protein Rations. Derived from Equation (27). Figures Indicate Per Cent Protein in Ration.

rations relatively higher in carbohydrates have higher marginal productivities than those high in protein. An interesting technical phenomenon is illustrated in the convergence point of the marginal product lines at a total feed input of about 4.2 pounds. While the predictions are based on the quadratic function (27), predictions from the other equations show similar relationships.

Predictions of isoquants from the overall power function (28) are provided in Figure 12. They show that predictions for the two variable functions give feed estimates corresponding closely to outcomes for the individual rations. (The dots show quantities from observations along single ration lines as they were fed in the experiment.) The isoquants in Figure 13 are from interval equations (29) and (30) and were used to predict least-cost rations as "averages for practical use" over the two weight ranges. Each of these isoquants is plotted with respect to the origin for the particular weight interval and not the feed quantities at the origin of both weight intervals (and thus are not successive contours on a production surface corresponding to accumulated feed inputs along the feed plane). This system of plotting also illustrates the declining marginal productivity of feed, since a larger amount of a given ration is required to produce a one-pound

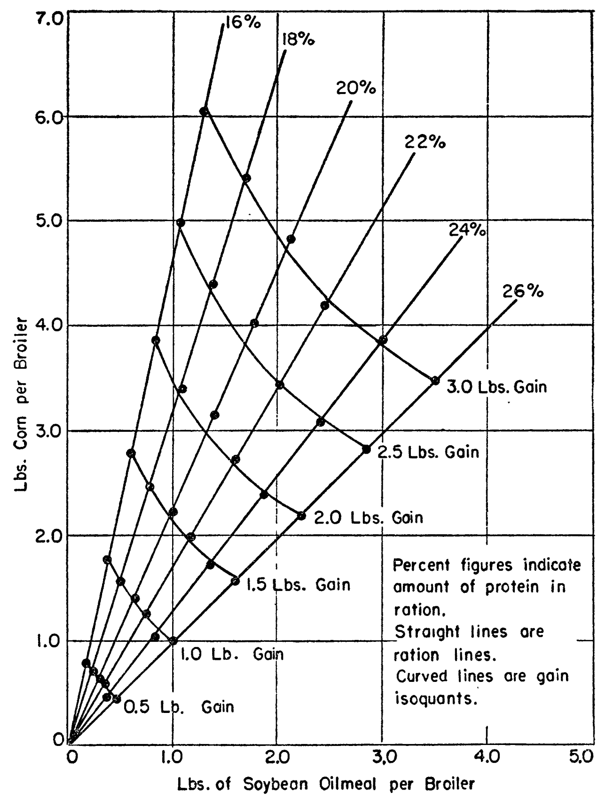

FIGURE 12.—Gain Isoquants Predicted from Equation (28). Dots Show Feed Quantities Along Single Variable Ration Lines as Fed.

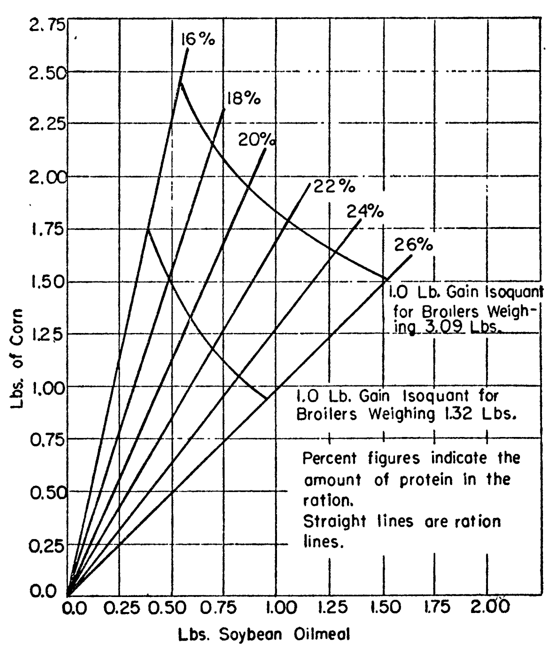

FIGURE 13.—One Pound Gain Isoquants for Broilers of 1.32 and 3.09. Pounds from Equations (29) and (30). Ration Lines with Per Cent Protein Indicated.

gain for heavier birds. The data illustrated, and other derived quantities, have been used in providing various types of guides for the economic decisions of broiler producers but are not included here because of space limitations.

4. SUMMARY

The data provided in this paper provide insights into (a) the nature of technical production functions and (b) economic decisions which rest on the algebraic form of the relationships. They illustrate the mutual gain to technologists and economists when funds which would otherwise be committed to less inclusive experiments are devoted to cooperative studies. They illustrate that econometric models stand to provide better information for economic decision-making than conventional biological models which assume discrete phenomena. Production functions estimated for fertilization, hog production, milk production, and poultry production have allowed derivation of relevant production surfaces, isoclines, isoquants, and the general set of marginal quantities which can be used with price ratios in specifying economic optima. While additional research is necessary to provide final knowledge of the exact biological nature of production functions, the data provided in this paper represent a large-scale start.

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