A Class of Variable Elasticity of Substitution Production Functions

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A CLASS OF VARIABLE ELASTICITY OF SUBSTITUTION PRODUCTION FUNCTIONS

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We introduce and analyze a class of variable elasticity of substitution (VES) production functions for which the substitution parameter varies linearly with the capital-labor ratio around the intercept term of unity. The VES function contains as special cases the more important special cases of the well known CES function. In terms of some familiar economic relationships, the VES posits a linear view of the world in contrast to the log-linear view posited by the CES function.

1. INTRODUCTION

ECONOMETRIC STUDIES involving the use of production functions often assume a specific numerical value for what has now become an important parameter, the elasticity of substitution parameter.[2] Such an assumption normally implies, of course, a unique functional form for the production function.[3] Thus the use of the popular Cobb-Douglas (CD) function implies that this parameter, to be denoted by σ, equals unity, while the use of the less popular fixed coefficient model and straight-line isoquant (the linear) production function implies that σ equals zero and infinity respectively. This parameter, however, can in principle assume any value between zero and infinity. This being so, any ad hoc assumption about the numerical value of σ can possibly lead to a specification bias. Recently, however, a new production function was proposed which eliminates this bias to the extent that the parameter σ is in fact a *constant*. This is the now famous constant elasticity of substitution (CES) production function (Arrow, Chenery, Minhas, and Solow [**2**], Brown and de Cani [**3**]) which allows the value for σ to be determined by the data.[4] Clearly, the CES function contains the above production functions as special cases.

In principle, however, the elasticity of substitution parameter σ can be a variable depending upon output and/or factor combinations (Hicks [**8**], Allen [**1**]), so that the assumption of a constant σ may contain a specification bias.

[1] This paper is based on my doctoral dissertation submitted to the University of Wisconsin, August, 1967. I am very much thankful to Professor Arnold Zellner but for whose encouragement in the initial stages, my thesis would not have been undertaken. Professor Arthur Goldberger, my thesis advisor, was all too kind to allow me to have free access to his valuable time at various points. I also wish to thank Professor Richard Day and especially, Professors Murray Brown and Zvi Griliches for their valuable comments. Comments from an unknown referee were also quite helpful. Errors remaining are of course my sole responsibility. The work was financed by the Ford Foundation Doctoral Dissertation Fellowship and the funds made available by the University of Wisconsin Research Committee. The empirical part of this paper was financed by research grants from the State University of New York at Buffalo, New York.

[2] For an elaborate account of these studies, see Walters [**25**].

[3] This statement is true, however, only if other parameters—the neutral efficiency parameter, the distribution parameter and the returns to scale parameter—are just parameters and not functions of any of the economic variables entering the production function. The statement, e.g., is not true if the returns to scale parameter is a function of the level of output (see Zellner and Revankar [**27**]).

[4] For empirical studies of the CES, see Walters [**25**].

Once we recognize that σ may be a variable, we at once face a variety of ways in which to choose the functional dependence of σ on output and/or inputs. Preferably, of course, this choice should be such that it permits testing of a constant σ, so that the production function corresponding to the choice made is a generalization of the CES, *in every respect*. But a main constraint on such a choice may be the desire for a convenient economic interpretation of the selected behavior of σ and for a resulting production function that is empirically manageable and economically insightful. Consider for example, the CES generalization suggested by Bruno [**4**] (and by Liu and Hildebrand [**9**]):

$$(1.1)\qquad V = \gamma[(1-\delta)K^{\eta} + \delta K^{m\eta}L^{(1-m)\eta}]^{1/\eta},$$

where V is output, K is capital, L is labor; γ, δ, m, and η are parameters.[5] The function in (1.1) reduces to the CES function when $m = 0$:

$$(1.2)\qquad V = \gamma[(1-\delta)K^{\eta} + \delta L^{\eta}]^{1/\eta}.$$

The elasticity of substitution for the function in (1.1) is given by

$$(1.3)\qquad \sigma = \frac{1}{1 - \eta + \dfrac{m\eta}{S_k}}$$

where S_k is capital's share. The function (1.1) is known to have an important economic implication[6] but it presents difficult nonlinearities for sound econometric methods of estimation.[7] Nor is the behavior of σ in (1.3) easily comprehensible.[8]

Of course, it remains to be seen if there exists a suitable choice of the functional form for σ that permits a convenient CES generalization. But in the meanwhile, we may seek some functional form for σ that permits a CD generalization.[9] In this instance, we may wish to insist that this CD generalization also contain the linear production function as a special case—the CD function itself contains the Harrod-Domar fixed-coefficient model as a special case and so will the CD generalization. The main rationale behind this insistence is that the resulting CD generalization will also contain most of the special cases of the CES function while at the same time providing for a variable σ.[10] It should then be of some

[5] For multi-factor generalizations of the CES, see e.g., Uzawa [**24**], and K. Sato [**16**].

[6] See Liu and Hildebrand [**9**]. The implication is the following:

$$(1.4)\qquad \log(V/L) = h_0 + h_1 \log w + h_2 \log(K/L)$$

where w is the wage rate and h_0, h_1, and h_2 are coefficients that are functions of the parameters of the function in (1.1).

[7] Of course, we do not mean to suggest here that the ease of econometric estimation of a production function is its prime virtue.

[8] As it is, the function does not seem to have received any attention from empirical workers. Indeed, its authors abandon it after a cursory treatment in their own work (Liu and Hildebrand [**9**]).

[9] The CD function is defined later in (1.6).

[10] The CD function does not contain the two-factor Leontief model, and this is true also of the CD generalization to be considered below (see (2.1)). On the other hand, the CES is known to include the Leontief two-factor model as a special case.

interest to compare the empirical relevance of the CD generalization and the CES function.

In this light, we take a look at a couple of rather familiar CD generalizations which do exhibit variable behavior of σ. Consider first the following function:

$$V = \gamma_1 e^{aK/L} K^{1-b} L^b \tag{1.5}$$

where V, K, and L are as before and γ, a, and b are parameters.[11] This function reduces to the CD function when $a = 0$, given by

$$V = \gamma_1 K^{1-b} L^b. \tag{1.6}$$

The function in (1.5), however, does not contain the linear function given by

$$V = aK + bL. \tag{1.7}$$

Moreover, it posits a considerably complex behavior for σ:

$$\sigma = \frac{(ax + 1 - b)(b - ax)}{(ax + 1 - b)(b - ax) - ax} \tag{1.8}$$

where $x = K/L$. Consider next, the transcendental production function (Halter, Carter, and Hocking [7]),

$$V = \gamma\, e^{a_1 K + a_2 L} K^{1-b} L^b. \tag{1.9}$$[12]

This function contains the CD function when $a_1 = a_2 = 0$, but it once again fails to contain the linear production function (1.7). Further, it too implies a rather involved behavior for σ:

$$\sigma = \frac{(1 - b + a_1 K)(b + a_2 L)}{[(1 - b)(b + a_2 L)^2 + b(1 - b + a_1 K)^2]}. \tag{1.10}$$

Lastly, there is the constant marginal share (CMS) function (Bruno [5]):

$$V = \gamma K^{\alpha} L^{1-\alpha} - mL \tag{1.11}$$

where γ, α, and m are parameters.[13] The CMS reduces to the CD function when $m = 0$. In contrast to the CD generalizations in (1.5) and (1.9), however, the CMS

[11] The function in (1.5) is a neoclassical production function only in a restricted region in the nonnegative orthant of the (K, L) plane, the region in which the marginal products are nonnegative and diminishing marginal rate of substitution holds. Thus, for example, if in (1.5) $a > 0$ and $0 < b < 1$, the marginal product of labor is nonnegative only if $b/a \geqslant K/L$.

[12] The transcendental function is a neoclassical function over the entire nonnegative orthant in the (K, L) plane only if $a_1 > 0$, $a_2 > 0$, and $0 < b < 1$. But now assume that $0 < b < 1$. If $a_1 > 0$ and $a_2 < 0$, the marginal product of labor is non-negative only if $L \leqslant b/-a_2$; the marginal product of capital in this case is always nonnegative. If $a_1 < 0$ and $a_2 > 0$, we should have, simply interchanging the role of capital and labor, $K \leqslant (1 - b)/-a_1$. In either case, we have diminishing marginal rate of substitution. If, however, $a_1 < 0$ and $a_2 < 0$, the neoclassical requirements are met only if $L \leqslant b/-a_2$ and $K \leqslant (1 - b)/-a_1$.

[13] The CMS function has nonnegative marginal product of labor over the entire (K, L) orthant (the marginal product of capital is always nonnegative under the usual assumption that $0 < \alpha < 1$) only if $m \leqslant 0$. But then the elasticity of substitution will always be equal to or greater than unity which would tend to limit the usefulness of the production function.

contains the linear production function.[14] What is more, it also exhibits a very convenient variable behavior for σ:

$$(1.12) \qquad \sigma = 1 - \left(\frac{m\alpha}{1 - \alpha}\right)\frac{L}{V}.$$

This behavior implies that as V/L increases (as in the context of economic development), σ tends to unity and the CMS converges to the CD function as a limit.[15] The CMS, however, awaits further empirical studies and rigorous econometric methods of estimating it.

In this paper we consider a specific CD generalization corresponding to the following choice of σ:

$$(1.13) \qquad \sigma = 1 + \beta\frac{K}{L}$$

where β is a parameter.[16] Equation (1.13) says that σ varies linearly with the capital-labor ratio. We call the production function corresponding to this σ, for lack of a better name, the variable elasticity of substitution (VES) production function. It is seen from (1.13) that $\sigma = 1$ when $\beta = 0$, i.e., the VES degenerates to the CD function. Thus, the test of the null hypothesis that $\beta = 0$ should be of considerable interest in assessing the empirical relevance of the VES function. Also, it will be seen below (see Section 2), that the VES includes some of the special cases of the CES, namely the Harrod-Domar fixed-coefficient model, the linear production function, and the CD function.[17] This calls for a comparative study of the empirical performance of the two functions, the VES and the CES.

[14] This is true only if m is negative when $\alpha = 1$. This shows that m should be suitably reparametrized so that the above condition is met. However, it is not at once clear how this can be accomplished.

[15] For further properties and an empirical application, see Bruno [**5**].

[16] The function in (1.13) is a special case of the function

$$(1.14) \qquad \sigma = \beta_0 + \beta\frac{K}{L}.$$

The production function corresponding to (1.14)—a CES generalization—was obtained late in 1965 (Revankar [**12**]) and independently of Sato's work [**17**, **18**]. This was done in the spirit of the "random coefficient" approach, a field of my early interest. Owing to the growing interest recently in the parameter σ, the latter was the obvious choice for consideration as a random variable. The specific choice of the form

$$\sigma = \beta_0 + \beta\frac{K}{L} + \varepsilon$$

rather than $\sigma = \sigma_0 + \varepsilon$ (a typical way of representing a parameter under the said approach, ε being a disturbance term) was made with a view to imparting economic content to the parameter. The motivation for Sato's work, on the other hand, derives from the empirical evidence in Wise and Yeh [**26**] that σ is "strongly correlated with the capital-labor ratio."

In the text we have considered the case where $\beta_0 = 1$, since no convenient expression obtains for the production function for $\beta_0 \neq 1$. It should be noted here that, with this value of β_0, we have eliminated the possibility of effecting a CES generalization.

[17] As noted in footnote 10, the VES *does not* include the special case of the Leontief (two-factor) fixed-coefficient model which the CES in fact does.

In this paper we are primarily interested in the theoretical aspects of the VES function. In Section 2, we discuss these, by way of bringing out succinctly the differences in the economic implications of the VES and the CES functions. As will be observed below, the most striking difference between these two functions is that the VES asserts *linear* relationships among important economic variables, while the CES posits *log-linear* relationships among these same variables. In Section 3, we report in a tongue-in-cheek manner, some empirical evidence on the VES function and its performance compared to that of the CES function. This tongue-in-cheek manner is called for here in view of the lack of data on the price of capital variable.[18] The findings reported here, therefore, are only tentative.

2. THE VES PRODUCTION FUNCTION: SOME THEORETICAL PROPERTIES

The VES Production Function

The VES production function to be introduced in this section is

$$V = \gamma K^{\alpha(1-\delta\rho)}[L + (\rho - 1)K]^{\alpha\delta\rho}, \qquad \gamma > 0, \quad \alpha > 0, \quad 0 < \delta < 1, \quad 0 \leqslant \delta\rho \leqslant 1, \quad \frac{L}{K} > \left(\frac{1-\rho}{1-\delta\rho}\right), \tag{2.1}$$

where V is output, K is capital, and L is labor; α, δ, ρ, and γ are parameters. It can be verified that the elasticity of substitution σ for the VES is (Revankar [**12**])

$$\sigma = \sigma(K, L) = 1 + \frac{\rho - 1}{1 - \delta\rho}\frac{K}{L}.^{19} \tag{2.2}$$

It is clear that we have taken $\beta = (\rho - 1)/(1 - \delta\rho)$. Thus the σ for the VES varies linearly with the capital-labor ratio around the intercept term of unity. We assume that $\sigma > 0$ in the empirically relevant range of K/L. This requires that $L/K > (1 - \rho)/(1 - \delta\rho)$ as spelled out in (2.1).

[18] We have developed elsewhere (Revankar [**13**]) a simultaneous equation system for the estimation of the VES function. There, the properties of the stochastic disturbances in the system are carefully specified and a sound econometric estimation procedure is sketched. This system, however, *works* only if the data on the price of capital variable are available. The lack of these data has of course plagued earlier empirical studies by other authors, and so has it in our case (Revankar [**13**, **14**]). While other authors have been bold enough to report on their empirical findings in an authentic manner, we have preferred here not to do so. In particular, we shall not discuss here the derivation of the VES simultaneous equation system, nor for that matter shall we discuss in detail the empirical results based on this system. This is because all our findings are based on the data on the price of capital variable generated according to $r = (V - W)/K$, where r is taken as the price of capital. And it is well known that such a procedure of generating the data on this variable raises too many econometric problems.

[19] It is a simple matter to obtain a production function whose σ varies linearly with the inverse of the capital-labor ratio, by simply interchanging the role of K and L.

Some Properties of the VES Function

First, the VES satisfies the requirements of a neoclassical production function:

$$(2.3)\qquad \begin{aligned}\frac{\partial V}{\partial L} &= \text{marginal product of labor}\\ &= \alpha\delta\rho\frac{V}{L+(\rho-1)K} > 0\end{aligned}$$

in view of the constraints of $0 \leqslant \delta\rho \leqslant 1$ and $L/K > (1-\rho)/(1-\delta\rho)$ spelled out in (2.1);

$$(2.4)\qquad \begin{aligned}\frac{\partial V}{\partial K} &= \text{marginal product of capital}\\ &= \alpha(1-\delta\rho)\frac{V}{K} + \alpha\delta\rho(\rho-1)\frac{V}{L+(\rho-1)K} > 0,\end{aligned}$$

and since $(\partial V/\partial K)/(\partial V/\partial L)$ is the marginal rate of substitution of capital for labor which equals $((\rho-1)/(1-\delta\rho)) + ((1-\delta\rho)/\delta\rho)(L/K)$,

$$(2.5)\qquad \frac{\partial\left[\dfrac{\partial V}{\partial K}\Big/\dfrac{\partial V}{\partial L}\right]}{\partial(K/L)} \leqslant 0.$$

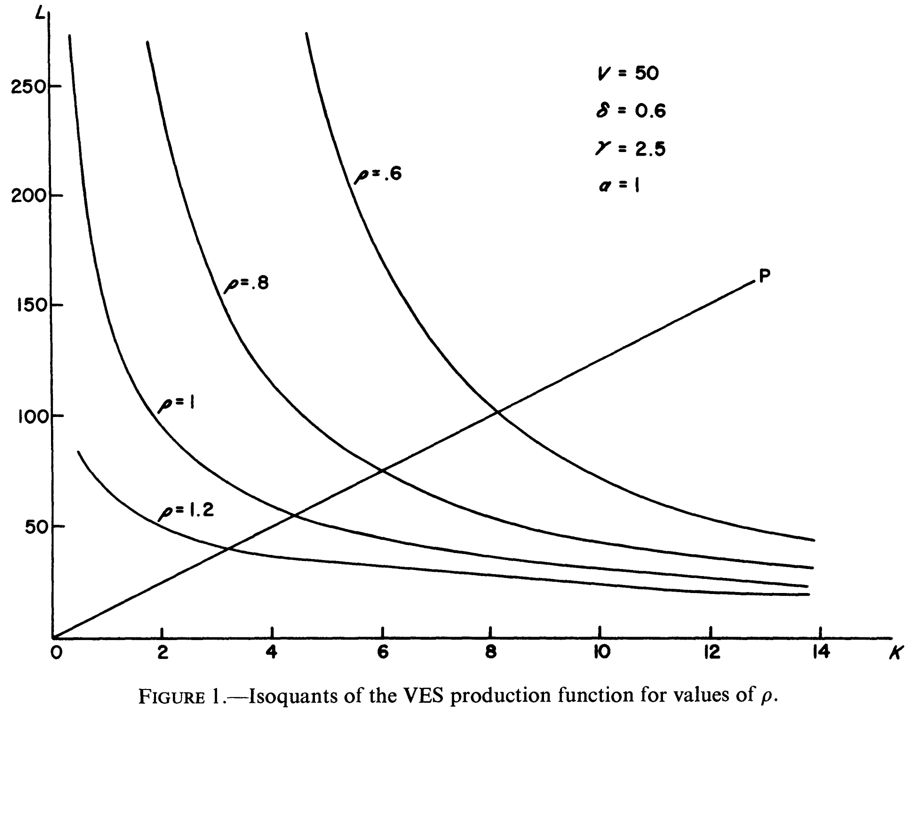

FIGURE 1.—Isoquants of the VES production function for values of ρ.

Second, it is clear from (2.1) that the VES includes the Harrod-Domar fixed-coefficient model ($\rho = 0$), the CD function ($\rho = 1$), and the linear production function ($\rho = 1/\delta\,(>1)$). Thus as ρ increases from zero to $1/\delta\ (>1)$, the elasticity of substitution increases steadily from zero to infinity.[20] In terms of the isoquants, this fact is expressed by their becoming more and more flattened as shown in Figure 1.[21]

Third, the VES differs from the CES in one important respect. The CES requires that the elasticity of substitution be the same at *all points of an isoquant*, independent of the level of output, hence at *all points of the isoquant map*. The VES, on the other hand, requires that this substitution parameter should be the same *only along a ray* (like *OP* in Figure 1); this parameter can vary along an isoquant.

The VES and the CES Functions: Their Implications

In this section and hereafter, we shall assume that there are constant returns to scale in the VES function, i.e., $\alpha = 1$. As it is, the CES in (1.2) posits constant returns to scale.

Assuming competitive conditions in the factor and the product markets, and profit-maximization, the marginal conditions are obtained by equating the marginal physical products of labor and capital to their real prices. Thus for the VES—see (2.3) and (2.4)—

$$\text{(2.6)} \qquad \delta\rho\frac{V}{L_1} = w,$$

and

$$\text{(2.7)} \qquad (1 - \delta\rho)\frac{V}{K} + \delta\rho(\rho - 1)\frac{V}{L_1} = r,$$

where w is the wage rate, r is the price of capital, and $L_1 = L + (\rho - 1)K$. Substituting from (2.6) into (2.7),

$$\text{(2.8)} \qquad \frac{V}{K} = F_0 r + F_1 w$$

where $F_0 = 1/(1 - \delta\rho)$ and $F_1 = (1 - \rho)/(1 - \delta\rho)$; and dividing (2.7) by (2.6) and using $L_1 = L + (\rho - 1)K$, we obtain

$$\text{(2.9)} \qquad \frac{L}{K} = G_0 + G_1 r/w,$$

where $G_0 = (1 - \rho)/(1 - \delta\rho)$ and $G_1 = \delta\rho/(1 - \delta\rho)$. For the CES (1.2), on the other hand, the marginal conditions comparable to (2.8) and (2.9) are known to be

$$\text{(2.10)} \qquad \log(V/K) = F'_0 + F'_1 \log r$$

[20] In fact, $d\sigma/d\rho = ((1 - \delta)(K/L)/(1 - \delta\rho)2) > 0$.

[21] Note that the isoquants in Figure 1 are drawn only over the range $L/K > (1 - \rho)/(1 - \delta\rho)$.

where $F_0' = (-\eta/(1-\eta))\log\delta - (1/(1-\eta))\log(1-\delta)$ and $F_1' = 1/(1-\eta) = \sigma$, and

$$\text{(2.11)} \qquad \log(L/K) = G_0' + G_1' \log(r/w)$$

where $G_0' = (1/(1-\eta))\log(\delta/(1-\delta))$ and $G_1' = 1/(1-\eta) = \sigma$. Now compare the two pairs of equations: (2.8)–(2.9) and (2.10)–(2.11). First of all, consider (2.8) and (2.10). Both purport to relate the output-capital ratio to factor prices. Equation (2.8) says that this ratio depends upon *both* the wage rate and the price of capital, being a *linear* function of the two. By contrast, equation (2.10) says that this ratio depends *only* on the price of capital, in a *log-linear* fashion. This linear versus log-linear relationship is even more sharply brought out by equations (2.9) and (2.11). Equation (2.9) says that the labor-capital ratio is a linear function of the relative price, r/w, while equation (2.11) asserts a log-linear relationship between these variables.[22]

This linear versus log-linear relationship, implied by the VES versus the CES function, also appears in the behavior of the relative income shares of labor and capital. Multiplying both sides of (2.9) by w/r,

$$\text{(2.12)} \qquad \frac{wL}{rK} = G_1 + G_0\frac{w}{r}.$$

Since $\alpha = 1$, wL is labor's income while rK is capital's income, where $rK + wL = V$, so that the income share of labor relative to that of capital, namely wL/rK is a linear function of w/r. On the other hand, as is well known or as can be easily verified from (2.11), the CES function implies that

$$\text{(2.13)} \qquad \log\left(\frac{wL}{rK}\right) = G_0' + (1 - G_1')\log(w/r),$$

which is a log-linear relationship between wL/rK and w/r.

Thus one obtains the impression that the VES subscribes to the view of a linear model while the CES sees it as a log-linear model.

This linear versus log-linear view of the world implied by the VES function and the CES function, respectively, has an important implication for the phenomenon of factor-intensity reversals between industries across countries. Minhas [**10**] in his admirable work demonstrated the empirical plausibility of such a phenomenon and relied for the purpose, on the CES relation (2.11). As against this relation, the VES asserts that the correct relation to use is given by (2.9). The empirical evidence on factor-intensity reversals is, as we know, based on the possibility that different industries may be characterized by different elasticities of substitution. This possibility can obviously be entertained within the framework of the VES function. What is more, since the capital-labor ratio is one of the variables involved, the issue may itself be sensitive to the behavior of σ

[22] Such relations, though not in these variables, have attracted attention in the literature both at the theoretical level (Theil [**20**]) and in the empirical context (Prais and Houthakker [**11**], Solow [**19**]). Which of these relations is the more plausible one is entirely an empirical question.

with this ratio. It follows then, that the VES, by allowing σ to vary with the capital-labor ratio, though in a simple-minded way, holds promise of shedding some new light on this delicate issue.

The Behavior of the Elasticity of Substitution in the VES Function

As noted in (2.2) the elasticity of substitution of the VES varies linearly with the capital-labor ratio, around the intercept term of unity. More specifically, when $\rho < 1$ (>1), it decreases (increases) steadily with this ratio and stays below (above) unity over the relevant (entire) range of the capital-labor ratio. Incidentally, it should be pointed out here that this one-sidedness in the behavior of σ (σ is either less or greater than unity over the sample) is a shortcoming of the VES function.[23]

From the behavior of σ of the VES, it is clear that in order to be able to effectively discriminate between the VES and CES functions, we need to have enough variation in the capital-labor ratios over the sample range. Such a happy situation, however, may be hard to encounter in practice, especially in the cross-section setup.

Finally, before we proceed to Section 3, we may pause to take a brief look at the "random coefficient" interpretation of $\sigma = \sigma(K, L)$ as defined in (2.2). The general stand of econometricians is that inputs are random variables endogenously determined. If so, K/L is a random variable distributed over the (hypothetical) population of firms, and the same applies to σ. Accordingly, assuming that K/L has a finite mean $\lambda\,(= E(K/L))$, we may define the *average* elasticity of substitution, $\bar{\sigma}$ say, *in the industry*, as

$$(2.14) \qquad \bar{\sigma} = E(\sigma(K, L)) = 1 + \frac{\rho - 1}{1 - \delta\rho}\lambda.$$

3. THE VES FUNCTION: SOME TENTATIVE FINDINGS

We have estimated elsewhere (Revankar [**14**]),[24] the VES function using cross-section data for the year 1957, on twelve selected two-digit United States manufacturing industries. The data were on a per-establishment basis and the states were the units of observation. The data on V, L, and W (wage bill) were taken from the *Annual Survey of Manufacturers, 1957* [**21**]. Capital figures correspond to the service-flow concept of capital. The service-flow of capital was defined as: depreciation and depletion charged in 1957 plus 0.06 times the net capital stock plus

[23] This shortcoming of the VES function obviously calls for a new production function for which σ crosses over from one side of unity to the other over the relevant range of the capital-labor ratio. We may mention here, however, that this crossover can be secured in the VES function in a time-series context where the VES is modified to incorporate biased technical change (in the Hicksian sense)—Revankar [**15**].

[24] We need to mention here that the results obtained in Revankar [**13**] contrast sharply with those indicated here. In the former, we had assumed arbitrary returns to scale (α) while here we have constrained α to equal unity. Thus the industries for which $\bar{\sigma}$ is significantly different from unity when α is arbitrary are not necessarily those for which $\bar{\sigma}$ is so when $\alpha = 1$. Further, with α arbitrary, there were instances where $\bar{\sigma}$, when significantly different from unity, was greater than 1.

insurance premiums, rental payments and property taxes paid.[25] Capital figures were computed from the data available in the 1958 *Census of Manufacturers, Vol. I, Summary Statistics*, Table 2 [**22**]. The data on r were generated residually according to $r = (V - W)/K$.

The test of the hypothesis $\sigma = 1$, performed by testing $G_0 = 0$ in (2.9), revealed the empirical relevance of the VES function in five out of twelve industries analyzed. In each of these five industries, the condition $(L/K) > (1 - \rho)/(1 - \delta\rho)$ required by the VES—see (2.1)—is met for each observation in the industry. Under constant returns to scale, average elasticity of substitution $\bar{\sigma}$ estimated at the observed mean capital-labor ratio was less than unity in all five cases.[26]

A less than unitary $\bar{\sigma}$ (or for that matter σ) means that σ in (2.2) decreases with the capital-labor ratio. One concludes therefore that as an industry gets more labor intensive, it experiences larger elasticities of substitution than when it gets more capital-intensive. And this is true uniformly for the "investment-oriented" and "consumer-oriented" industries.

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[25] Net capital stock was defined as: "gross book value on December 31, 1957" minus "accumulated depreciation and depletion charged in 1957." This definition of the capital service flow was taken from Griliches [**6**].

[26] The reader should be reminded here that the findings given below are only tentative. They contain an unknown amount of bias because the "price of capital" variable was generated residually, making it an endogenous variable. Since this variable enters on the right-hand side of (2.9) on which our results are based, the results are biased. The estimate of σ, in particular, is likely to contain a bias towards zero in view of the transitory fluctuations in profits as well as the errors of measurement in K.

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