Diagnosability of Hypercubes and Enhanced Hypercubes under the Comparison Diagnosis Model

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Abstract—In [10], Sengupta and Dahbura discussed how to characterize a diagnosable system under the comparison diagnosis model proposed by Maeng and Malek [7] and a polynomial algorithm was given to identify the faulty processors provided that the system's diagnosability is known. However, for a general system, the determination of its diagnosability is not algorithmically easy. This paper proves that, for the important hypercube-structured multiprocessor systems (n-cubes), the diagnosability under the comparison model is n when $n \geq 5$. The paper also studies the diagnosability of enhanced hypercube [11], which is obtained by adding 2^{n-1} more links to a regular hypercube of 2^n processors. It is shown that the augmented communication ability among processors also increases the system's diagnosability under the comparison model. We will prove that the diagnosability is $n+1$ for an enhanced hypercube when $n \geq 6$.

Index Terms—Diagnosability, diagnosis by comparison, graph theory, hypercube, interconnection network.

1 Introduction

The hypercube structure is a well-known interconnection model for multiprocessor system. As a topology to interconnect processors, it has many attractive properties. Multicomputer systems built with hypercube structure have already been commercially available [2], [3] [4], [9]. Because of its importance to achieving high performance, the fault-tolerant computing for hypercube structures has been the interest of many researchers. Generally speaking, fault tolerance is achieved either by providing spare processors or by computing in the presence of faulty processors. No matter which strategy of the two is used, as the first step to deal with faults, the system should be able to discriminate the faulty processors from the fault-free ones. The process of determining faulty processors is called the *diagnosis* of the system.

Several different approaches have been developed for interconnected processors to diagnose faulty processors among themselves. One major approach, first proposed by Malek and Maeng [6], [7], performs the diagnosis by sending the same input to pairs of processors and comparing their responses. Based on the collective results of comparisons, one may be able to claim the faulty/fault-free status of the processors in the system. Since its proposal, the diagnosis-by-comparison method has drawn extensive interests of researchers. A paper by Sengupta and Dahbura [10] gives a thorough study of this approach. It reveals important properties of diagnosable systems under this approach; several characterizations of diagnosable systems are given; and a polynomial algorithm is presented to determine the faulty processors in a general system provided the system is diagnosable.

In this paper, we study the diagnosability of hypercubes and one of its variants, the so-called enhanced hypercubes [11], under the diagnosis-by-comparison approach. Since the determination of diagnosability for a general system is not an easy task, it would be nice if we knew the diagnosability of some widely used systems, such as hypercubes. We will prove that the diagnosability of hypercubes is n under the diagnosis-by-comparison approach if $n \geq 5$. The enhanced hypercube is obtained by adding more links to the regular hypercube. These extra links increase the system's diagnostic ability. We will show that the diagnosability of the enhanced hypercubes is increased to $n+1$ under the diagnosis-by-comparison approach if $n \geq 6$. The rest of this paper is organized as follows: In Section 2, we give necessary backgrounds and definitions. In Section 3, we prove the n-diagnosability of regular hypercubes of dimension n. In Section 4, we prove the $(n+1)$-diagnosability of enhanced hypercubes. We then give some concluding remarks in Section 5.

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2 Preliminaries

In the study of multiprocessor systems, the topology of a system is often adequately represented by a graph $G = (V, E)$, where each node $v_i \in V$ represents a processor and each edge $\{v_i, v_j\} \in E$ represents a communication channel (link) between v_i and v_j. In the well-known PMC diagnostic model [8], based on the graph representation, the two linked processors can directly test each other. According to the system's overall test results, an assertion about the faulty/fault-free status of processors may be made. Numerous results using the PMC model have been derived since its first proposal in 1967.

In the comparison-based approach, first introduced by Maeng and Malek [7], the diagnosis is carried out by sending a certain input to a pair of processors and comparing their outputs. The comparison is performed by a third processor that has access to both processors being compared. This third processor can be adequately called the *comparator* of the former two. The result of the comparison is either that the two outputs agree or that they disagree. We always assume that two processors can access each other only if there is a link between them. So, processor v_a being able to perform comparison for v_b and v_c implies that $\{v_a, v_b\} \in E$ and $\{v_a, v_c\} \in E$.

The comparison scheme of the whole system can be modeled with a labeled multigraph $M = (V, C)$. V represents the set of processors. A labeled edge $\{v_b, v_c\}_{v_a} \in C$, with v_a being a label on the edge, connects v_b and v_c, which means that processors v_b, v_c are being compared by v_a. M is a multigraph because the same pair of nodes may be compared by different comparators. We use $r(\{v_b, v_c\}_{v_a})$ to represent the result of comparing processors v_b and v_c by v_a such that

$$r\left(\{v_b, v_c\}_{v_a}\right) = \begin{cases} 0 & \text{if outputs agree} \\ 1 & \text{if outputs disagree} \end{cases}$$

Clearly, if $r(\{v_b, v_c\}_{v_a}) = 1$, then at least one of v_a, v_b, v_c must be faulty. If $r(\{v_b, v_c\}_{v_a}) = 0$ and v_a is known to be fault-free, then v_b, v_c are all fault-free. (We assume, quite reasonably, that the input is designed in such a way that it is extremely unlikely that v_b, v_c with both or one being faulty would produce the same result.) If the comparator is faulty, then the result of comparison is unreliable so that no conclusion can be drawn of the status of v_b and v_c. Fig. 1 gives an example graph G and a comparison multigraph M in which a pair of nodes are compared by a third one wherever it is possible. Notice that the multigraph is not unique. Different multigraphs can be obtained for the same G. The collective result of all comparisons, formally defined as a function $s : C \rightarrow \{0, 1\}$, is called the *syndrome* of the diagnosis. By analyzing a syndrome, we may or may not be able to make assertions about the faulty/fault-free status of the processors in the system. A subset $F \subseteq V$ is said to be *consistent* with a syndrome s if s can arise from the circumstance that all nodes in F are faulty and all nodes in $V - F$ are fault-free. It is important to point out that, for a given syndrome s, there may be more than one faulty subset of V that are consistent with s. If this happens, the system G with comparison scheme M cannot diagnose for syndrome s since the faulty-sets that can cause s are not unique. A system is called *diagnosable system* if, for *every* syndrome s, there is a

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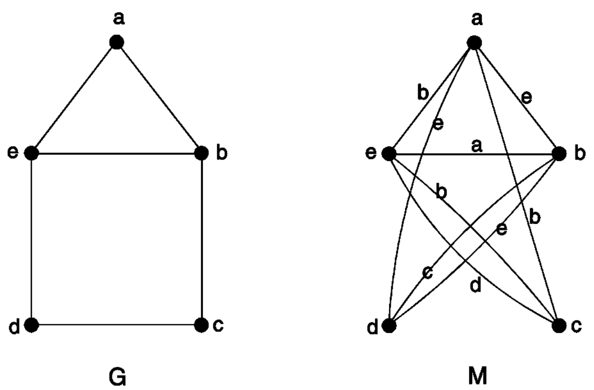

Fig. 1. A graph G and its one possible comparison multigraph M.

unique $F \subseteq V$ that is consistent with s. It is clear that, for a system to be able to self-diagnose, there should be an upper bound for the number of faulty processors. We define the *diagnosability* of a system to be a number t such that, as long as the number of faulty processors is less than or equal to t, the system is diagnosable. We call such a system a t-diagnosable system.

As can be seen, different comparison schemes can be established for a fixed system. To gain as much knowledge as possible about the faulty status of the system, one can do as many comparisons as one can. That is, a comparison is conducted for *every* 3-tuple v_a, v_b, v_c such that $\{v_a, v_b\}, \{v_a, v_c\} \in E$, with v_a being the comparator. The multigraph representing this special comparison scheme can be formally defined as follows:

$$M^* = (V, C^*) \text{ where } C^* = \{\{v_b, v_c\}_{v_a} \mid \{v_a, v_b\}, \{v_a, v_c\} \in E\}.$$

Notice that as, soon as the system $G = (V, E)$ is given, the M^* is known. In the rest of the paper, we will assume that this particular comparison scheme is adopted for the given system.

The *n-hypercube* is a well-known interconnection model. An n-hypercube can be viewed as a graph $G = (V, E)$ such that V consists of 2^n nodes, numbered from

$$\underbrace{00\ldots0}_{n}$$

to

$$\underbrace{11\ldots1}_{n}.$$

$\{v_i, v_j\} \in E$ if and only if v_i and v_j have only one bit different. Thus, every node has links with exactly n other nodes. There are,

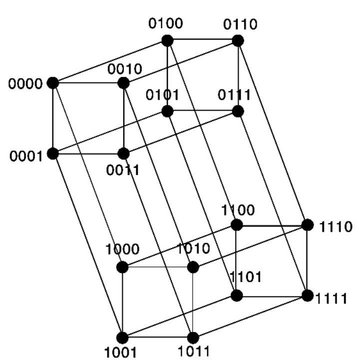

Fig. 2. A 4-dimensional hypercube, or 4-cube for short.

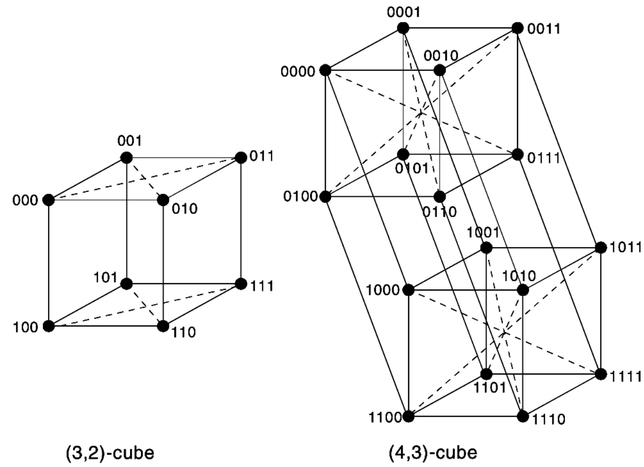

Fig. 3. $(3,2)$-cube and $(4,3)$-cube. The skips are shown with dashed links.

altogether, $n2^{n-1}$ links. Two nodes v_i, v_j of an n-hypercube that have d bits different are said to have *Hamming distance d*, denoted as $H(v_i, v_j) = d$. So, in an n-hypercube, a link exists between v_i and v_j if and only if $H(v_i, v_j) = 1$. Fig. 2 gives an example of 4-hypercube. From now on, we will call n-hypercube simply n-cube. Notice that the nodes can be numbered differently as long as the above link regulation is obeyed. Also notice that an $(n+1)$-cube is constructed by linking the corresponding nodes of two n-cubes. Letting Q_{n+1} denotes an $(n+1)$-cube, we denote the two n-cubes forming Q_{n+1} as

$$Q_n^0 = 0\underbrace{xx..x}_{n}$$

and

$$Q_n^1 = 1\underbrace{xx..x}_{n},$$

respectively. n-cube as a topology to interconnect processors has many attractive properties. It is a very important architecture for multiprocessor systems, and several commercial multicomputer systems using n-cube structure are already on the market [2], [3], [4], [9]. The diagnosability of hypercubes under other diagnostic models was studied [1], [5]. An n-cube under the PMC model was shown to be n-diagnosable [1]. In Section 3 of this paper, we will prove that, under the comparison model, an n-cube is also n-diagnosable.

In this paper, the diagnosability of *enhanced hypercubes* under the comparison model is also studied. In real multiprocessor systems using hypercube structure, the processors are often manufactured with the maximum allowable links. In most cases, not all these links are used to implement the regular hypercube of the system. This gives the motivation for utilizing the leftover links to get extra connections between the nodes. These hypercubes with extra links can be called enhanced hypercubes. We call these extra links *skips*. It was shown that this augment in the system's communication ability can notably improve the system performance as a whole [11]. In [11], Tzeng and Wei investigated the performance of one category of enhanced hypercubes, in which 2^{n-1} skips are added in the following way: In addition to the $n2^{n-1}$ regular links in an n-cube, a skip exists between a pair of nodes $b_n b_{n-1} \ldots b_{k+1} b_k b_{k-1} \ldots b_1$ and $b_n b_{n-1} \ldots b_{k+1} \bar{b}_k \bar{b}_{k-1} \ldots \bar{b}_1$, where $k \in [2, \ldots, n]$ is the Hamming distance between the two nodes linked by a skip. We use notation (n,k)-*cube* to denote an enhanced n-cube with parameter k. Examples of $(3,2)$-cube and $(4,3)$-cube are given in Fig. 3. The proposal of having extra links is a very practical one because these links are equipped as the processors are fabricated, would be unused otherwise, and the implementa-

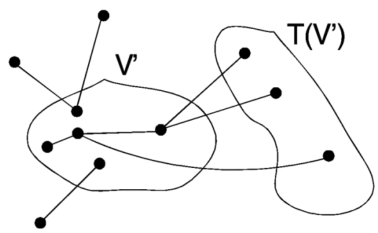

Fig. 4. An example of V' and its $T(V')$.

tion does not pose difficulty. It turns out that the enhanced hypercubes can achieve noticeable improvements in many measurements, such as mean internode distance, diameter, and traffic density, compared to regular hypercubes [11]. In Section 4, we will show that, for enhanced hypercubes, the diagnosability is also increased under the comparison model. More specifically, under the comparison model, the diagnosability of an enhanced n-cube is increased to $n+1$ when $n \geq 6$.

3 Diagnosability of Hypercubes under the Comparison Model

We need some more definitions for the discussions that follow. Let $N(v) = \{u \mid \{v,u\} \in E\}$, the set of all nodes that are linked to v. For a subset $V' \subset V$,

$$N(V') = \left(\bigcup_{v \in V'} N(v) \right) - V',$$

the set of nodes in $V - V'$ that are linked to nodes of V'.

Given a system $G = (V, E)$ with comparison scheme represented by labeled multigraph $M = (V, C)$, for a node $v_i \in V$, let X_{v_i} be the set of nodes that are connected to v_i by an edge either in E or in C. That is, a node in X_{v_i} is either linked to v_i or compared with v_i by some other node. Formally,

$$X_{v_i} = \{v_j \mid \text{ either } \{v_i, v_j\} \in E \text{ or } \{v_i, v_j\}_{v_h} \in C \text{ for some } v_h\}.$$

Let Y_{v_i} be the set of edges among nodes of X_{v_i} such that

$$Y_{v_i} = \{\{v_j, v_h\} \mid v_j, v_h \in X_{v_i} \text{ and } \{v_i, v_j\}_{v_h} \in C\}.$$

Define $G_{v_i} = (X_{v_i}, Y_{v_i})$.

A *vertex cover* of a graph $G = (V, E)$ is a subset $K \subseteq V$ such that every edge of E has at least one end in K. Obviously, there could be vertex covers of different cardinalities. A vertex cover of minimum cardinality is called *minimum vertex cover*. For a node $v_i \in V$, define the *order of* v_i to be the cardinality of a minimum vertex cover of G_{v_i}.

Given system G and the comparison scheme M, for a subset of nodes $V' \subset V$, let $T(V')$ denote the set of nodes that are outside of V' and are compared to some node of V' by some node of V'. Formally,

$$T(V') = \{v_j \mid \{v_i, v_j\}_{v_h} \in C \text{ and } v_i, v_h \in V' \text{ and } v_j \notin V'\}.$$

Fig. 4 illustrates a simple example of $T(V')$.

There are several different ways to characterize a t-diagnosable system under the comparison approach [10]. For a general graph, none of these characterizations is algorithmically easy. The determination of diagnosability for a general graph is actually intractable. For our purpose, we will use one particular characterization given by Sengupta and Dahbura which gives a sufficient condition for a system to be t-diagnosable.

Theorem 1 [10]. *A system with N nodes is t-diagnosable if 1) $N \geq 2t+1$, 2) each node has order at least t, 3) for each $V' \subset V$ such that $|V'| = N - 2t + p$ and $0 \leq p \leq t-1$, $|T(V')| > p$.*

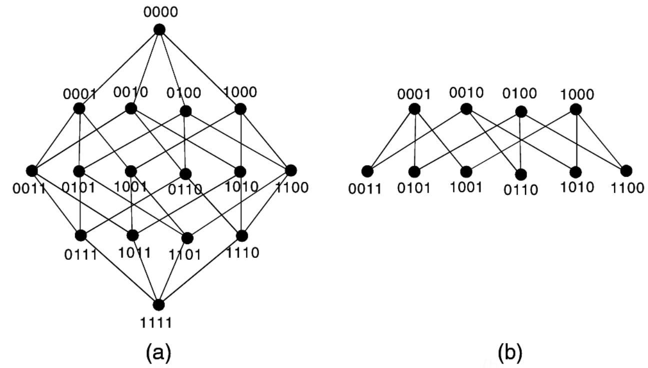

Fig. 5. (a) A 4-cube drawn in layered manner and (b) its G_{0000}.

We will prove that an n-cube is n-diagnosable by proving that it satisfies the sufficient condition of Theorem 1.

Lemma 1 [5]. *For any two nodes u, v in n-cube, $|N(\{u,v\})| \geq 2n-2$.*

Lemma 2. *For any three nodes u, v, w in n-cube, $|N(\{u,v,w\})| \geq 3n-5$.*

Proof. We prove the lemma by induction on n, the dimension of hypercube.

Basis. When n is small, the claim can be checked by inspection.

Hypothesis. The claim holds for n-cube.

Induction. Consider an $(n+1)$-cube, Q_{n+1}. Q_{n+1} is composed of two n-cubes

$$Q_n^0 = \underbrace{0xx..x}_{n+1}$$

and

$$Q_n^1 = \underbrace{1xx..x}_{n+1}$$

such that each node $0b_n \dots b_1$ in Q_n^0 is linked to $1b_n \dots b_1$ in Q_n^1.

If u, v, w all fall in Q_n^0, then, by hypothesis, u, v, w have at least $3n-5$ neighbors, all in Q_n^0. But, u, v, w have three more neighbors in Q_n^1. Therefore,

$$|N(\{u,v,w\})| \geq (3n-5) + 3 = 3(n+1) - 5.$$

Suppose now u, v fall in Q_n^0, w falls in Q_n^1. By Lemma 1, u, v have at least $2n-2$ neighbors, all in Q_n^0. w will bring in at least n new neighbors in Q_n^1. Therefore,

$$|N(\{u,v,w\})| \geq (2n-2) + n = 3(n+1) - 5.$$

□

Lemma 3. *For any four nodes u, v, w, x in n-cube,*

$$|N(\{u,v,w,x\})| \geq 4n-9.$$

Proof. Again the lemma is proven by induction on n.

Induction. If u, v, w, x all fall in Q_n^0, then, by hypothesis, u, v, w, x have at least $4n-9$ neighbors in Q_n^0. But, u, v, w, x have four more neighbors in Q_n^1. Therefore,

$$|N(\{u,v,w,x\})| \geq (4n-9) + 4 = 4(n+1) - 9.$$

If u, v, w fall in Q_n^0, x falls in Q_n^1, then, by Lemma 2, u, v, w have at least $3n-5$ neighbors in Q_n^0. w will bring in at least n new neighbors in Q_n^1. Therefore,

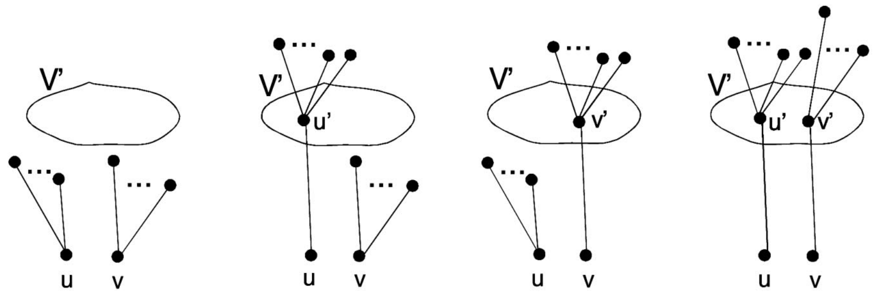

Fig. 6. For $u, v \notin T(V')$, one of the above four cases must hold.

$$|N(\{u, v, w, x\})| \geq (3n-5) + n = 4(n+1) - 9.$$

If u, v fall in Q_n^0, w, x fall in Q_n^1, then, by Lemma 1, u, v have at least $2n-2$ neighbors in Q_n^0, w, x have at least $2n-2$ neighbors in Q_n^1. Therefore,

$$|N(\{u, v, w, x\})| \geq (2n-2) + (2n-2) > 4(n+1) - 9.$$

□

Lemma 4. *A node of an n-cube has order n.*

Proof. Since all the nodes of an n-cube link to others in exactly the same way, it suffices to check the order only for node $v_i = 0 \ldots 00$. By the definition of $G_{v_i} = (X_{v_i}, Y_{v_i})$, X_{v_i} consists of those nodes that are either linked to v_i or being compared with v_i. So, X_{v_i} is the union of two sets:

$$X_{v_i} = \{\text{all nodes with one 1}\} \quad (\text{linked to } v_i)$$
$$\cup \{\text{all nodes with two 1s}\} \quad (\text{being compared with } v_i).$$

Y_{v_i} consists of all edges $\{v_j, v_h\}$ such that v_h is a comparator of v_i and v_j, i.e., v_h is linked to v_i and v_j is linked to v_h. That is,

$$Y_{v_i} = \{\{v_j, v_h\} \mid v_h \text{ has one 1}, v_j \text{ has two 1s}\}.$$

For the convenience of observation, we depict, in Fig. 5, an example n-cube in a "layered" manner and the corresponding G_{v_i}. It can be seen that G_{v_i} is a symmetric-structured, so-called bipartite graph (a graph whose vertices can be partitioned into two sets V_1 and V_2 such that every edge has one end in V_1 and the other in V_2). The minimum vertex cover of G_{v_i} can be determined simply by inspection, which is the set of the nodes having one 1. It has cardinality n, which is the order of v_i by definition. □

We are now ready to state and prove the theorem about the diagnosability of n-cubes under the comparison model.

Theorem 2. *An n-cube-structured system, represented by graph $G = (V, E)$, with V being the node set and E the link set, is n-diagnosable under the comparison model if $n \geq 5$ and the comparison scheme is $M^* = (V, C^*)$.*

Proof. We prove that an n-cube satisfies the sufficient condition for n-diagnosability in Theorem 1.

The first condition, $2^n \geq 2n + 1$, is trivially true when $n \geq 3$. The second condition is satisfied by Lemma 4. It remains to show that an n-cube satisfies the third condition, i.e., for an arbitrary $V' \subset V$ such that $|V'| = 2^n - 2n + p$, $0 \leq p \leq n-1$, we want to show $|T(V')| > p$. We give the definition of $T(V')$ here again for the reader's convenience:

$$T(V') = \{v_j \mid \{v_i, v_j\}_{v_h} \in C \text{ and } v_i, v_h \in V' \text{ and } v_j \notin V'\}.$$

We first show that the claim is true for $p = n-1$. We then prove that the claim also holds for $p = 0, 1, \ldots, n-2$.

If $p = n-1$, then $|V'| = 2^n - n - 1$, giving

$$|V - V'| = n + 1. \tag{1}$$

For the sake of contradiction, assume $|T(V')| \leq n-1$, meaning that $V - V'$ contains at least two nodes which are not in $T(V')$. Let $u, v \in V - V'$ such that $u, v \notin T(V')$. From the definition of $T(V')$, if $u \notin T(V')$, then u must have the following property: Either $N(u) \cap V' = \phi$, or if $u' \in N(u) \cap V'$, then $N(u') \cap V' = \phi$. For $u, v \notin T(V')$, one of the four cases illustrated in Fig. 6 must hold.

Notice that, in every case of the four, $V - V'$ will contain at least all neighbors of two nodes—in the first case, u and v; in the second case, u' and v; in the third case, u and v'; in the fourth case, u' and v' ($u' \neq v'$ because if $u' = v'$, we are in the second case). By Lemma 1, two nodes have at least $2n-2$ neighboring nodes. But, $2n - 2 > n + 1 = |V - V'|$ if $n \geq 4$, a contradiction to (1).

For $p = 0, \ldots, n-2$, let $|V'| = 2^n - 2n + p$, giving

$$|V - V'| = 2n - p. \tag{2}$$

Again, for the sake of contradiction, assume $|T(V')| \leq p$. That means $V - V'$ contains at least $(2n - p) - p = 2(n - p)$ nodes that are not in $T(V')$. Since $p \leq n - 2$, we have $2(n-p) \geq 4$. Let $u, v, w, x \in V - V'$ be four nodes such that $u, v, w, x \notin T(V')$. For these four nodes, we can have a figure similar to Fig. 6, with 16 different cases, in each of which $V - V'$ would contain at least all neighbors of four nodes. By Lemma 3, four nodes have at least $4n - 9$ neighboring nodes. But, $4n - 9 > 2n - p = |V - V'|$ if $n \geq 5$. Again, a contradiction to (2). □

We point out that 5-cube is the least hypercube satisfying the sufficient condition. Fig. 7 gives an example 4-cube that does not satisfy condition 3) of Theorem 1.

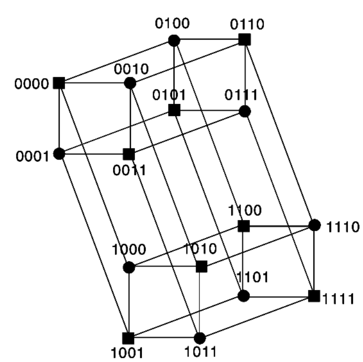

Fig. 7. A 4-cube. V' consists of all the square nodes. $|V'| = 2^4 - 2 \times 4 + 0 = 8$ ($p = 0$). Since none of the circle nodes belongs to $T(V')$, $|T(V')| > p$ does not hold.

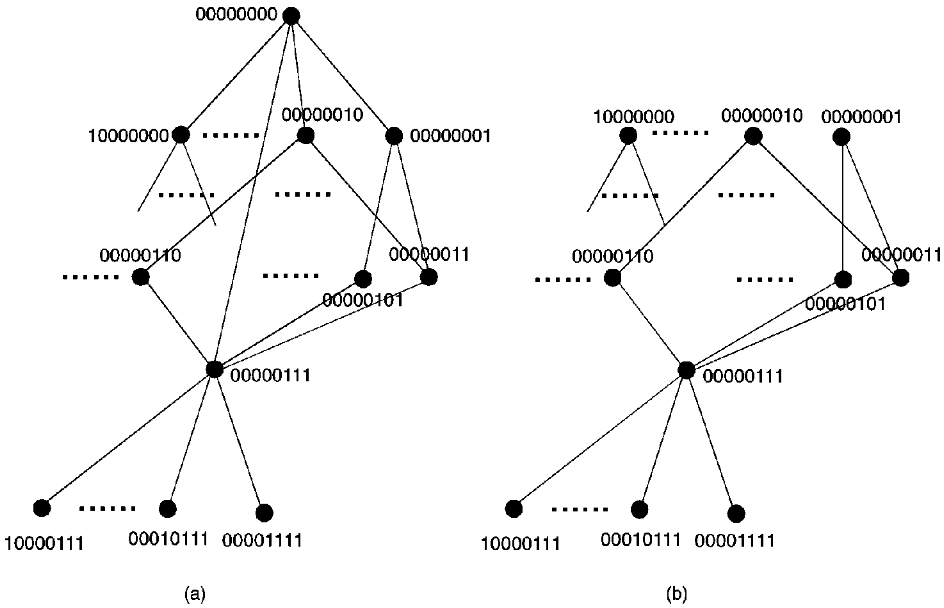

Fig. 8. (a) Part of an enhanced n-cube ($n = 8, k = 3$) drawn in layered manner and (b) its $G_{0..00}$.

Knowing the n-diagnosability of an n-cube under the comparison model, we can directly apply the diagnosis algorithm proposed in [10] to find the faulty processors of the n-cube, provided the number of faulty processors does not exceed n.

4 Diagnosability of Enhanced Hypercubes under the Comparison Model

Enhanced hypercubes can achieve noticeable improvements in many measurements, such as mean internode distance, diameter, and traffic density, compared to regular hypercubes [11]. In [12], it was proven that enhanced hypercubes can also achieve increased diagnosability over regular hypercubes under the PMC model. In this section, we will show that, for enhanced hypercubes, the diagnosability is also increased (to $n+1$) under the comparison model.

Lemma 5. *A node of an enhanced n-cube has order $n+1$, where $n \geq 4$.*

Proof. The proof takes a similar line to that of Lemma 4. We first construct $G_{v_i} = (X_{v_i}, Y_{v_i})$ for node $v_i = 0\ldots00$, taking into account that v_i has a skip to node $v_i' = 0..0\underbrace{1..1}_{k}$,

$$\begin{aligned} X_{v_i} = &\{\text{all nodes with one } 1\} && (\text{linked to } v_i)\\ &\cup \{\text{all nodes with two 1s}\} && (\text{being compared with } v_i)\\ &\cup \{v_i'\} && (\text{linked to } v_i)\\ &\cup \{\text{all nodes linked to } v_i'\} && (\text{being compared with } v_i)\\ Y_{v_i} = &\{\{v_h, v_j\} \mid v_h \text{ has one } 1, v_j \text{ has two 1s}\}\\ &\cup \{\{v_i', v_j'\} \mid v_i' = 0..0\underbrace{1..1}_{k}, \{v_i', v_j'\} \in E \text{ and } v_j' \neq v_i\}. \end{aligned}$$

An example enhanced n-cube and its corresponding $G_{0..00}$ are depicted in Fig. 8. It can be observed that the minimum vertex cover of G_{v_i} consists of all the nodes having one 1 plus the node

$$v_i' = 0..0\underbrace{1..1}_{k}.$$

Notice that, if $n = 3$ and $k = 2$, a minimum vertex cover could be obtained by taking all the nodes having two 1s (node v_i' already included). Since the numbers of two-1-nodes and one-1-nodes are both n when $n = 3$, the minimum vertex cover has cardinality n. For $n \geq 4$, the number of two-1-nodes is always greater than that of one-1-nodes, so it is impossible to have a vertex cover of cardinality n. Hence, the cardinality of the minimum vertex cover is $n+1$ when $n \geq 4$. □

Theorem 3. *An enhanced n-cube-structured system, represented by graph $G = (V, E)$, with V being the node set and E the link set, is $(n+1)$-diagnosable under the comparison model if $n \geq 6$ and the comparison scheme is $M^* = (V, C^*)$.*

Proof. The proof is similar to that of Theorem 2. We prove by showing that an enhanced n-cube satisfies the sufficient condition for $(n+1)$-diagnosability.

The first condition is true when $n \geq 3$. The second condition, that every node has at least order $n+1$ (here, $n \geq 6$), is satisfied by Lemma 5. We will show in the rest of the proof that an enhanced n-cube satisfies the third condition of Theorem 1. That is, we want to show that, for an arbitrary $V' \subset V$ such that $|V'| = 2^n - 2(n+1) + p$, $0 \leq p \leq n$, we have $|T(V')| > p$. We first show that the claim is true for $p = n$. We then show that the claim also holds for $p = 0, 1, \ldots, n-1$.

When $p = n$, $|V'| = 2^n - 2(n+1) + n = 2^n - n - 2$, resulting in

$$|V - V'| = n + 2. \tag{3}$$

For the sake of contradiction, assume $|T(V')| \leq n$, meaning that $V - V'$ contains at least two nodes which are not in $T(V')$. Let $u, v \in V - V'$ such that $u, v \notin T(V')$. From the definition of $T(V')$, if $u \notin T(V')$, then either $N(u) \cap V' = \phi$, or if $u' \in N(u) \cap V'$, then $N(u') \cap V' = \phi$. For $u, v \notin T(V')$, one of the four cases illustrated in Fig. 6 (in the last section) must hold. In either of the four cases, $V - V'$ will contain at least all neighbors of two nodes. By Lemma 1, two nodes have at least $2n-2$ neighboring nodes. But, $2n - 2 > n + 2 = |V - V'|$ if $n \geq 5$, a contradiction to (3). Therefore, $|T(V')| > n$ must hold.

For $p = 0, \ldots, n-1$, let $|V'| = 2^n - 2(n+1) + p$, giving

$$|V - V'| = 2n + 2 - p. \tag{4}$$

Again, for the sake of contradiction, assume $|T(V')| \leq p$. That means $V - V'$ contains at least $(2n + 2 - p) - p = 2(n + 1 - p)$

nodes that are not in $T(V')$. Since $p \leq n-1$, we have $2(n+1-p) \geq 4$. Let $u, v, w, x \in V - V'$ be four nodes such that $u, v, w, x \notin T(V')$. For these four nodes, we can have a figure similar to Fig. 6, with 16 different cases, in each of which $V - V'$ would contain at least all neighbors of four nodes. By Lemma 3, four nodes have at least $4n - 9$ neighboring nodes. But, $4n - 9 > 2n + 2 - p = |V - V'|$ for $n \geq 6$. Again, a contradiction to (4). Therefore, $|T(V')| > p$ must hold. □

We have just shown that an enhanced n-cube-structured system has improved diagnosability over a regular n-cube system when $n \geq 6$. Having known the $(n+1)$-diagnosability under the comparison model, one can apply the diagnosis algorithm proposed in [10] to find the faulty processors of the enhanced n-cube. This time, the system can allow up to $n+1$ faulty processors to perform the diagnosis. As the manufacturing technology advances rapidly, the failure probability of each processor drops considerably. Therefore, the increase of diagnosability by even one, so achieved in enhanced n-cube, will greatly extend the system's overall self-diagnostic ability.

5 Conclusion

The diagnosability of hypercubes and one of its variants, the enhanced hypercubes, under the comparison diagnosis model is studied in this paper. It has been shown that the n-dimensional hypercube is n-diagnosable under the comparison model. For the enhanced hypercube, which is obtained by adding 2^{n-1} more links to a regular hypercube, the diagnosability is $n+1$. Since it is usually a hard task to determine the diagnosability for a general system under the comparison model, it is hoped that we know the diagnosability of some widely-used systems, the hypercube being one of them. Knowing the diagnosability of a hypercube-structured system, we can directly apply the algorithm proposed in [10] to diagnose an n-cube, provided that there will be at most n faulty processors, or diagnose an enhanced n-cube, provided that there will be at most $n+1$ faulty processors.

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