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The diagnosability of hypercubes with arbitrarily missing links

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Abstract

We study the problem of determining diagnosability for incomplete hypercubes that have arbitrarily distributed missing links, under the classic PMC diagnostic model and its variant, the BGM model. Based on the result proved in this paper, for both models, in most cases the diagnosability of an incomplete hypercube can be determined by simply checking the link degree of each node. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Many topologies have been proposed to interconnect processors in a multiprocessor computer system. Among them, the *hypercube* has drawn the greatest attention. A hypercube possesses many attractive properties and has become the most popular architecture for multiprocessor systems. Commercial multiprocessor systems based on hypercube structure have already been available [3,5,6,13]. Because of its importance for achieving high performance, the fault-tolerant computing for hypercubes has been the interest of many researchers.

It is well known that the diagnosability of an n-dimensional hypercube is n under the classic PMC model [11]. That is, an n-dimensional hypercube can correctly detect all faulty nodes provided that the number of faulty nodes does not exceed n. When the adopted fault bound is maximum, n, all the links will be involved in the diagnosis. Algorithms are available to determine the faulty processors provided that the diagnosability of the system is known [1,9]. We can apply diagnosis algorithms only when we know the diagnosability of the system. Having diagnosability n in an n-dimensional hypercube implies that all links among nodes are functioning, i.e., there are no missing links. It is then a natural question to ask how the diagnosability decreases if some links are missing. The past literature in this field has seen studies of diagnosability for hypercubes with regularly enhanced links [15] or regularly decreased links [17]. In this paper, we present results that establish the diagnosability for hypercubes in the presence of *arbitrarily* distributed missing links. We will give a simple algorithm to decide the diagnosability for incomplete hypercubes with missing links (failing, or in use other than diagnosis).

The diagnosability of incomplete hypercubes under a variant of PMC model, the BGM model [2], is also studied in this paper. A simple algorithm is presented that decides the diagnosability for incomplete hypercubes under the BGM model. As an immediate corollary of the main results in

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this part, we will show that the diagnosability of a complete n-dimensional hypercube is also n under the BGM model.

The rest of this paper is organized as follows. In Section 2, we give necessary backgrounds for our problems, definitions and previous results. Section 3 is on the diagnosability of incomplete hypercubes under PMC model and is in two parts. The first part proves the main mathematical results our algorithm will be based on. The second part presents the algorithm and analyzes its complexity. Section 4 addresses the same problem for BGM model. Section 5 gives some concluding remarks.

2. Preliminaries and previous results

In the study of multiprocessor systems, the structure of a system is usually adequately represented by a graph $G = (V, E)$, where each node $v_i \in V$ represents a processor and each edge $\{v_i, v_j\} \in E$ represents a communication channel between v_i and v_j. An *n-dimensional hypercube*, *n-cube* for short and denoted as Q_n, is a graph $G = (V, E)$ such that V consists of 2^n nodes, numbered from $\underbrace{00\ldots0}_{n}$ to $\underbrace{11\ldots1}_{n}$, and an edge (or link) $\{v_i, v_j\} \in E$ if and only if v_i and v_j have exactly one bit different. Thus, each node has immediate links with, and therefore can directly access to, exactly n other nodes. It is easy to establish that $|E| = n2^{n-1}$. Two nodes v_i, v_j of an n-cube that have d bits different are said to have *Hamming distance d*, denoted as $H(v_i, v_j) = d$. So in an n-cube, a link exists between v_i and v_j if and only if $H(v_i, v_j) = 1$. Fig. 1 shows a Q_3 and a Q_4. It can be seen that an $(n+1)$-cube is made up by two n-cubes, with node $\underbrace{0b_n\ldots b_1}_{n+1}$ in one n-cube being linked to $\underbrace{1b_n\ldots b_1}_{n+1}$ in the other. It is convenient to denote the two *subcubes* as $Q_n^0 = \underbrace{0xx\ldots x}_{n+1}$ and $Q_n^1 = \underbrace{1xx\ldots x}_{n+1}$, respectively. Since any two nodes linked by an edge have one and only one bit different, an edge can be uniquely represented using the two nodes it links. If $v_i = b_n..b_k..b_1$, $v_j = b_n..\bar{b}_k..b_1$, then we use $b_n..b_{k+1}Xb_{k-1}..b_1$ to denote edge $\{v_i, v_j\}$. We call $b_n..b_{k+1}Xb_{k-1}..b_1$ an edge of dimension k. Every node is incident to exactly n edges, one edge in each dimension. There are totally 2^{n-1} edges in each dimension. For instance, in a 3-cube, node 000 has 3 edges incident to it: $00X$, $0X0$ and $X00$. The four edges of dimension 2 are $0X0, 0X1, 1X0$ and $1X1$.

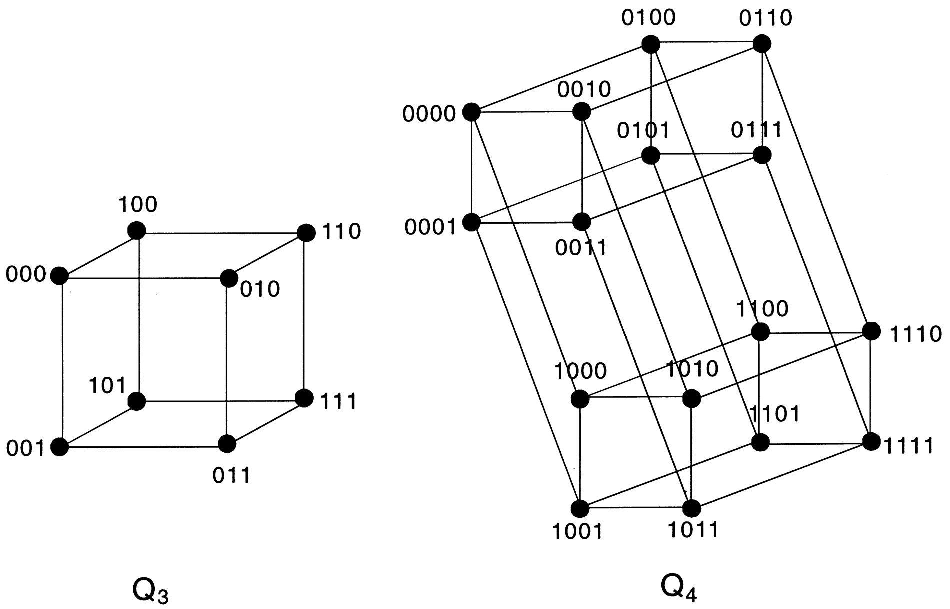


Fig. 1. A 3-cube and a 4-cube.

In the well-known PMC multiprocessor diagnostic model, the self-diagnosable system is represented by a *directed graph* $G = (V, A)$, or digraph for short, in which a node v_i can test all nodes v_j if arrow $(v_i \to v_j) \in A$. An undirected graph $G = (V, E)$ is a special digraph in which $(v_i \to v_j) \in A \leftrightarrow (v_j \to v_i) \in A$, meaning that two linked nodes can perform tests on each other. We use $\Gamma(v)$ to represent all nodes that v can test, $\Gamma^{-1}(v)$ to represent all nodes that can test v. For a subset of $V' \subset V$, we define

$$\Gamma V' = \bigcup_{v \in V'} \Gamma(v) - V' \text{ and } \Gamma^{-1} V = \bigcup_{v \in V'} \Gamma^{-1}(v) - V'.$$

Regardless of the specific test methods, the final test result is simply a conclusion that the tested node is "faulty" or "fault-free," denoted as label 1 or 0 on the corresponding arrow. The PMC model assumes that a fault-free node should always give correct test result, whereas the test result given by a faulty node is unreliable, i.e., it will be arbitrarily 0 or 1. The collection of all test results, called a *syndrome*, can be formally defined as a function $s : A \to \{0, 1\}$. A subset $F \subseteq V$ is said to be *consistent* with a syndrome s if s can arise from the circumstance that all nodes in F are faulty and all nodes in $V - F$ are fault-free. It is worth pointing out that for a given syndrome s, there may be more than one subset of V that are consistent with s. If this happens, the system cannot diagnose for syndrome s, because the faulty-sets that can cause s are not unique. It is clear that under the PMC model, there must be some (at least one) good processors to correctly perform diagnosis. For a given system, the *diagnosability* is an integer t such that if $|F| \leqslant t$, the diagnosis can be guaranteed to carry out correctly. Formally,

Definition 1. A system is said to be t-diagnosable (or has diagnosability t) if for any syndrome s, there is one and only one faulty-subset $F \subseteq V$ that is consistent with s, given that the number of faulty nodes does not exceed t.

Later, a variant of the PMC model, named BGM model after its proposers, was introduced in Ref. [2]. The difference between the two models is the following. In the PMC model, a test result 0 or 1 from v_i testing v_j is reliable only if v_i (the tester) is fault-free. In other words, a 0 test result does not necessarily mean that the tested node is fault-free. However, in the BGM model, a test result 0 from v_i testing v_j must mean that v_j is fault-free. The rationale for this assumption is that the tests are usually complex and extensive enough so that if either one or both of v_i and v_j are faulty, it is extremely unlikely that a 0 result will come about. For the BGM model, the definition of diagnosability is the same as in Definition 1.

In a hypercube-structured system, two linked processors can directly access each other and therefore can be managed to perform tests on each other. If $G = (V, E)$ represents the structure of the hypercube system and $\{v_i, v_j\} \in E$, then $v_i \in \Gamma^{-1}(v_j)$ and $v_j \in \Gamma^{-1}(v_i)$. The testing assignment is therefore the same as the topology of the system structure. The following theorem gives the PMC diagnosability of a *complete* n-cube, i.e., the n-cube with *all* its links participating in the diagnosis.

Theorem 1 ([1,8]). *A system of n-cube structure is n-diagnosable if* $n \geqslant 3$.

This paper is on determining the diagnosability for an n-cube with incomplete links. The same problem under both PMC and BGM models are studied. There are many reasons why we should be interested in determining the diagnosability when some links are not participating in the diagnosis. For example, there may be link failures in the system. Or we may want to use only part of the links to perform the diagnosis so that some normal computation and fault diagnosis can be carried out at the same time. All these scenarios require the n-cube to diagnose in the presence of some missing links. It is necessary for an n-cube to know its diagnosability (which is not n anymore) before the diagnosis takes place.

The following definitions and earlier results will be used in the rest of the paper.

Definition 2. The connectivity $\kappa(G)$ of a graph $G(V, E)$ is the minimum number of nodes whose removal results in a disconnected or a trivial (one node) graph.

Lemma 1 ([16]). *$\kappa(G) = K$ if and only if the least number of disjoint paths between any pair of nodes in G is K.*

Lemma 2 ([8]). *There exist exactly n disjoint paths between any two nodes in an n-cube.*

Lemma 3 ([11]). *Let $G = (V, E)$ be the graph representation of a system such that $|V| = N$. The necessary conditions for the system to be PMC t-diagnosable are*:
1. $N \geqslant 2t + 1$;
2. $\forall v \in V[|\Gamma^{-1}(v)| \geqslant t]$.

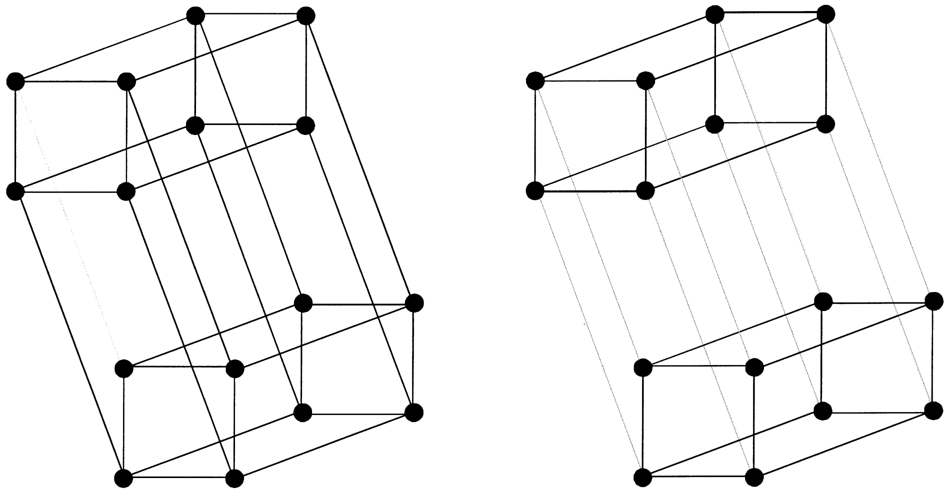
Lemma 4 ([4]). *Let $G = (V, E)$ be the graph representation of a system such that $|V| = N$. The sucient conditions for the system to be PMC t-diagnosable are*:
1. $N \geqslant 2t + 1$;
2. $\kappa(G) \geqslant t$.

Notice that Theorem 1 can be immediately obtained from Lemmas 1, 2 and 4.

3. Diagnosability algorithm for incomplete hypercubes under PMC model

For a general digraph, establishing its diagnosability is not easier than finding the faults. The first algorithm to determine the diagnosability for an arbitrary digraph which runs in polynomial time was proposed in Ref. [14]. The algorithm's complexity is $O(|A||V|^{1.5})$. Later an improved algorithm was suggested in Ref. [12], which runs in $O(|V|\tau^{2.5})$, where τ is the system's diagnosability. In the latter algorithm, the higher the system's diagnosability, the more running time it takes to determine it. Since τ is usually a far less number than $|A|$, the algorithm of Ref. [12] outperforms that of Ref. [14]. However, if the system is less general, e.g. incomplete n-cube in our case, we hope to altogether avoid using the general algorithm so that higher efficiency and simpler implementation can be achieved.

The distribution of missing links in the hypercube greatly affects the resulting (decreased) diagnosability. With even 1 link missing from an n-cube, shown in Fig. 2(a), the necessary condition in Lemma 3 cannot be satisfied anymore, so that the diagnosability will decrease by 1. However, if



(a) (b)

Fig. 2. The light lines represent missing links. (a) 1 link missing, $(n-1)$-diagnosable. (b) 2^{n-1} links missing, $(n-1)$-diagnosable.

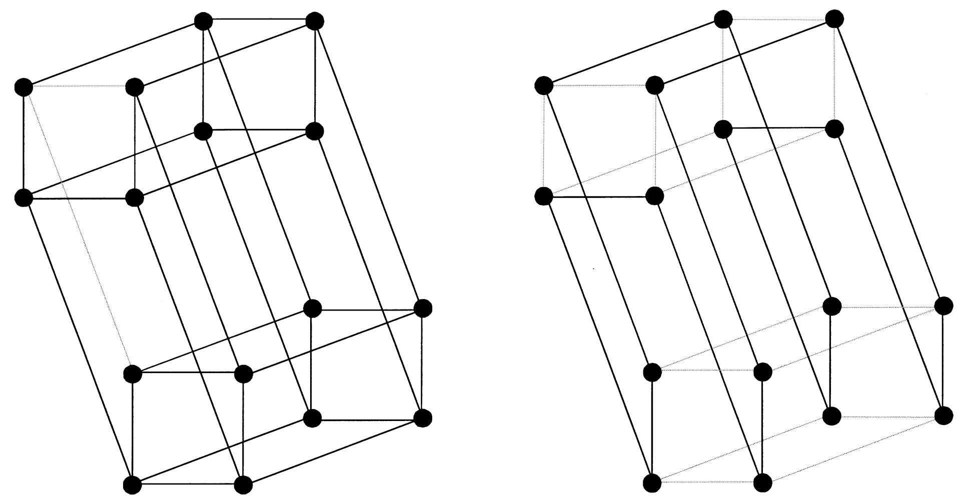


Fig. 3. Both systems are $(n-2)$-diagnosable. One has 2 missing links, while the other has $2 \times 2^{n-1}$ $(n=4)$.

more links in the same dimension are removed, the diagnosability does not decrease further. In the extreme case, all 2^{n-1} links in a dimension are taken away so that we have two disconnected $(n-1)$-cubes shown in Fig. 2(b). Even though the system is disconnected, its diagnosability remains $n-1$ if $n \geqslant 4$ (which can be readily proved with Lemmas 1, 2 and 4). Another example in Fig. 3: if only 2 links incident to the same node are missing, the diagnosability will immediately drop by 2. However, we can have as many as $2 \times 2^{n-1}$ links removed and the system remains $(n-2)$-diagnosable.

Since there are $n2^{n-1}$ links, there will be $2^{n2^{n-1}}$ different distributions of missing links. Fortunately, by some properties we will reveal in this section, in most cases, we do not have to look at the missing links' distributions.

3.1. The foundation of the algorithm

The characterization of a t-diagnosable system was given in Ref. [12], which our algorithm will be based on.

Lemma 5 ([12]). *A system $G = (V, A)$ is t-diagnosable if and only if*

$$\forall V' \subseteq V \left[V' \neq \phi \;\Rightarrow\; \frac{|V'|}{2} + |\Gamma^{-1}V'| > t \right].$$

For an n-cube with *complete* links, the following two lemmas are about the cardinality of test-set for a two- and three-node subset.

Lemma 6 ([7]). *For any two nodes u, v in n-cube, $|\Gamma^{-1}\{u, v\}| \geqslant 2n-2$.*

Lemma 7. *For any three nodes u, v, w in n-cube, $|\Gamma^{-1}\{u, v, w\}| \geqslant 3n-5$.*

Proof. We prove the lemma by induction on n, the dimension of hypercube.

Basis. When n is small, the claim can be checked by inspection.

Hypothesis. The claim holds for n-cube.

Induction. Consider an $(n+1)$-cube, Q_{n+1}. Q_{n+1} is composed of two n-cubes $Q_n^0 = \overbrace{0xx..x}^{}_{n+1}$ and $Q_n^1 = \overbrace{1xx..x}^{}_{n+1}$ such that each node $0b_n \ldots b_1$ in Q_n^0 is linked to $1b_n \ldots b_1$ in Q_n^1.

If u, v, w all fall in Q_n^0, then by Hypothesis, u, v, w have at least $3n-5$ testers, all in Q_n^0. But u, v, w have 3 more testers in Q_n^1. Therefore $|\Gamma^{-1}\{u, v, w\}| \geqslant (3n-5)+3 = 3(n+1)-5$.

Suppose now u, v fall in Q_n^0, w falls in Q_n^1. By Lemma 6, u, v have at least $2n-2$ testers, all in Q_n^0. w will bring in at least n new testers in Q_n^1. Therefore $\Gamma^{-1}\{u, v, w\}| \geqslant (2n-2)+n = 3(n+1) -5$. □

In the following theorem, $d(v)$ denotes the degree of node v. In an undirected graph, $|\Gamma^{-1}(v)| = d(v)$.

Theorem 2. *Given an n-cube system with incomplete links, graphically denoted as* $G=(V,E)$. *If* $\min\{d(v) \mid v \in V\} = r$ *such that* $r \geqslant 3$, *then the system's diagnosability is r under the PMC model.*

Proof. We prove the theorem by showing that for any non-empty subset V' of V, $|V'|/2 + |\Gamma^{-1}V'| > r$ will be satisfied. Then by Lemma 5, the system is r-diagnosable.

When $|V'| \geqslant 2r+1$, $|V'|/2 + |\Gamma^{-1}V'| > r$ will always hold. So we only have to show that the condition is satisfied for all the V's such that $|V'| \leqslant 2r$. For the sake of convenience we denote $|V'|/2 + |\Gamma^{-1}V'|$ as $\Phi(V')$.

Case 1 ($|V'| = 1$): For an arbitrary one-node $V' = \{v\}$, since $d(v) \geqslant r$, we have $\Phi(V') = 1/2 + d(v) > r$.

Case 2 ($|V'| = 2$): Let $V' = \{u, v\}$. By Lemma 6, $|\Gamma^{-1}V'| \geqslant 2n-2$ for a complete n-cube. Now that the n-cube is incomplete, some links from u and v may be missing. Let k_u (k_v) be the number of missing links from u (v). Since $d(u), d(v) \geqslant r$, $k_u, k_v \leqslant n-r$ must hold. Therefore $|\Gamma^{-1}V'| \geqslant (2n-2) - 2(n-r) = 2r-2$. We have $\Phi(V') \geqslant 2/2 + (2r-2) = 2r-1 > r$ if $r > 1$.

Case 3 ($|V'| = 3$): Let $V' = \{u, v, w\}$. By Lemma 7, $|\Gamma^{-1}V'| \geqslant 3n-5$ for a complete n-cube. Now that the n-cube is incomplete, some links from u, v and w may be missing. Let k_u (k_v, k_w) be the number of missing links from u (v, w). Since $d(u), d(v), d(w) \geqslant r$, $k_u, k_v, k_w \leqslant n-r$ must hold. Therefore $|\Gamma^{-1}V'| \geqslant (3n-5) - 3(n-r) = 3r-5$. So we have $\Phi(V') \geqslant 3/2 + (3r-5) = 3r-3.5 > r$ if $r > 1.75$.

Case 4 ($4 \leqslant |V'| \leqslant 2r$): Observe that the addition of one node into V' will decrease the value of $\Phi(V')$ at most by 1/2: adding one node into V' will increase $|V'|/2$ by 1/2; if this new node of V' has been chosen from the old $\Gamma^{-1}V'$, then the worst case is that $|\Gamma^{-1}V'|$ will be decreased by 1, resulting an overall decrease of $\Phi(V')$ by 1/2. So for a V' such that $4 \leqslant |V'| \leqslant 2r$, the lowest value we can get for $\Phi(V')$ is

$$\Phi(V') \geqslant (3r-3.5) - (2r-3) \times \frac{1}{2} = 2r-2 > r \quad \text{if } r > 2,$$

where $(3r-3.5)$ is the least Φ value for the first 3 nodes, and $(2r-3) \times 1/2$ is the worst decrease of Φ's value caused by additional nodes to V'.

Summarizing the preceding four cases, $\Phi(V') = |V'|/2 + |\Gamma^{-1}V'| > r$ when $|V'| \leqslant 2r$, provided that $r \geqslant 3$. For $|V'| > 2r$, $|V'|/2 + |\Gamma^{-1}V'| > r$ trivially holds. We arrive at the conclusion that $|V'|/2 + |\Gamma^{-1}V'| > r$ will be satisfied for all non-empty V' if $r \geqslant 3$. □

We point out that 3 is the least r that is equal to the system's diagnosability. Two example n-cube systems with incomplete links are given in Fig. 4, in both of which $\min d(v) = 2$, while one is 2-diagnosable, the other is not. So for $\min d(v) \leqslant 2$, the diagnosability cannot be simply decided just by $\min d(v)$. More intricate methods are in order, which are available from earlier research [14,12].

Also notice that Theorem 2 is a more generalized presentation of the well-known Theorem 1. From Theorem 2 we can immediately have the following corollary,

Corollary 1. *An n-cube system is n-diagnosable,* $n \geqslant 3$, *under the PMC model.*

3.2. The algorithm

Based on Theorem 2, and the PMC diagnosability algorithm in Ref. [12], we present the diagnosability algorithm for hypercubes with incomplete links. Suppose the incomplete hypercube under consideration is represented by a graph $G=(V,E)$, with V being the node set and E the (incomplete) link set.

Step 1. For every node, acquire the number of links incident to it that can be used for diagnostic

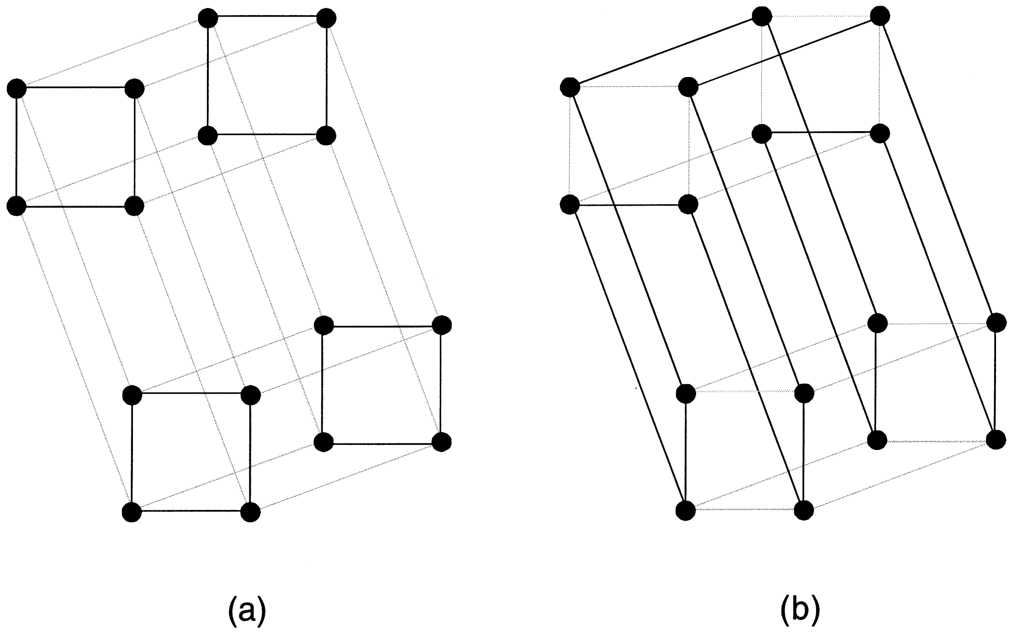


Fig. 4. (a) $d(v)=2$, not 2-diagnosable (it is actually 1-diagnosable). (b) $d(v)=2$ and 2-diagnosable.

task. Compute $r = \min\{$ *number of links incident to* $v \mid v \in V\}$.

Step 2. If $r \geqslant 3$, the diagnosability of the system is r and we are done. Otherwise, goto step 3.

Step 3. $r \leqslant 2$. We use the PMC diagnosability algorithm in Ref. [12] to determine the diagnosability of the system.

As the manufacturing technology progresses, the probability of link failure should continue to drop. So it can be anticipated that in most cases of link failure we have $r \geqslant 3$, meaning that the relatively complex algorithm in Step 3 is rarely invoked.

Since the validity of Theorem 2 and the algorithm of Ref. [12] has been established, the correctness of the above algorithm is self-evident. We now analyze the algorithm's running time. Step 1 takes $O(|V|)$ steps, supposing that each node can determine the failing/functioning status of its links. Step 2 is a constant operation. For Step 3, the algorithm of Ref. [12] is bounded by $O(|V|\tau^{2.5})$ [12], where τ is the system's diagnosability. Since $\tau \leqslant 2$ if Step 3 is reached, we have that Step 3 is $O(|V|)$. Summarizing the preceding argument, the diagnosability algorithm for incomplete hypercubes has complexity $O(|V|)$. As we have just pointed out, one should really anticipate $r \geqslant 3$ in most cases, so that determining diagnosability, without reaching Step 3, is a very simple process.

4. Diagnosability algorithm for incomplete hypercubes under BGM model

There have existed algorithms that compute the BGM diagnosability for a general testing system [10,12]. The algorithm in Ref. [10] has complexity $O(|V|\tau^3)$, where τ is the system's diagnosability. Raghavan and Tripathi [12] proposed an improved algorithm that effected an $O(|V|\tau^2/\log\tau)$ complexity. Just like in the case of PMC model, for a less general system such as incomplete n-cube, we wish to avoid using the general algorithm.

Lemma 8 ([10]). *If* $\min\{|\Gamma^{-1}(v)| \mid v \in V\} = r$, *then the diagnosability under the BGM model is either* r *or* $r-1$.

Let $V_r = \{v \mid |\Gamma^{-1}(v)| = r\}$. We define a binary relation $\equiv$ over V_r such that $u \equiv v$

$\Longleftrightarrow$ $\Gamma^{-1}(u) = \Gamma^{-1}(v)$. It is obvious that $\equiv$ is an equivalence relation so that it induces nodes of V_r into several equivalence classes.

Lemma 9 ([12]). *The diagnosability under the BGM model is* $r-1$ *if and only if there exist distinct equivalence classes* A *and* B *over* V_r *such that* $\Gamma^{-1}A - B = \Gamma^{-1}B - A$.

Theorem 3. *Given an n-cube system with incomplete links, graphically denoted as* $G = (V,E)$. *If* $\min\{d(v) \mid v \in V\} = r$ *such that* $r \geqslant 3$, *then the system's diagnosability is r under the BGM model.*

Proof. We prove the theorem by showing that in an incomplete hypercube, if $\min\{d(v) \mid v \in V\} \geqslant 3$, there will be no distinct equivalence classes A and B over V_r such that $\Gamma^{-1}A - B = \Gamma^{-1}B - A$. Then by Lemmas 8 and 9, the system's diagnosability is r.

Firstly, we show that every equivalence class contains only one node. Without loss of generality, we suppose $v_0 = 0\ldots000 \in V_r$ and show that $v_0 = 0\ldots000$ is an equivalence class by itself. Since any node can be numbered as $0\ldots000$, this proves that every node of V_r is an equivalence class by itself. We have

$$\Gamma^{-1}(v_0) \subseteq \{0\ldots001, 0\ldots010, 0\ldots100, \ldots, 1\ldots000\},$$

such that $|\Gamma^{-1}(v_0)| = r$. For any node $v_1 \in V_r$ such that $H(v_0, v_1) = 1$, we can always number $v_1 = 0\ldots001$. We have

$$\Gamma^{-1}(v_1) \subseteq \{0\ldots000, 0\ldots011, 0\ldots101, \ldots, 1\ldots001\},$$

such that $|\Gamma^{-1}(v_1)| = r$. It can be seen that $\Gamma^{-1}(v_0)$ and $\Gamma^{-1}(v_1)$ will have nothing in common, thus $\Gamma^{-1}(v_0) \neq \Gamma^{-1}(v_1)$. For any node $v_2 \in V_r$ such that $H(v_0, v_2) = 2$, we can always number $v_2 = 0\ldots011$. We have

$$\Gamma^{-1}(v_2) \subseteq \{0\ldots010, 0\ldots001, 0\ldots111, \ldots, 1\ldots011\},$$

such that $|\Gamma^{-1}(v_2)| = r$. It can be seen that $\Gamma^{-1}(v_0)$ and $\Gamma^{-1}(v_2)$ will have at most two nodes in common. Since $|\Gamma^{-1}(v_0)| = |\Gamma^{-1}(v_2)| = r \geqslant 3$, we have $\Gamma^{-1}(v_0) \neq \Gamma^{-1}(v_2)$. For any node $v_i \in V_r$ such that $H(v_0, v_i) \geqslant 3$, every node of $\Gamma^{-1}(v_i)$ will have at least two 1's. Therefore $\Gamma^{-1}(v_0) \neq \Gamma^{-1}(v_i)$.

We now show that there will be no distinct equivalence classes A and B over V_r, such that $\Gamma^{-1}A - B = \Gamma^{-1}B - A$. Let $A = \{v_0\}$. For $B = \{v_1\}$,

$$\begin{aligned}\Gamma^{-1}&A - B\\ &\subseteq \{0\ldots001, 0\ldots010, 0\ldots100, \ldots, 1\ldots000\}\\ &\quad - \{0\ldots001\}\\ &\subseteq \{0\ldots010, 0\ldots100, \ldots, 1\ldots000\},\end{aligned}$$

$$\begin{aligned}\Gamma^{-1}&B - A\\ &\subseteq \{0\ldots000, 0\ldots011, 0\ldots101, \ldots, 1\ldots001\}\\ &\quad - \{0\ldots000\}\\ &\subseteq \{0\ldots011, 0\ldots101, \ldots, 1\ldots001\}.\end{aligned}$$

Therefore $\Gamma^{-1}A - B \neq \Gamma^{-1}B - A$. For $B = \{v_2\}$,

$$\begin{aligned}\Gamma^{-1}&A - B\\ &\subseteq \{0\ldots001, 0\ldots010, 0\ldots100, \ldots, 1\ldots000\}\\ &\quad - \{0\ldots011\},\end{aligned}$$

$$\begin{aligned}\Gamma^{-1}&B - A\\ &\subseteq \{0\ldots010, 0\ldots001, 0\ldots111, \ldots, 1\ldots011\}\\ &\quad - \{0\ldots000\}.\end{aligned}$$

It can be clearly seen that $\Gamma^{-1}A - B = \Gamma^{-1}A$, and $\Gamma^{-1}B - A = \Gamma^{-1}B$. $\Gamma^{-1}A$ and $\Gamma^{-1}B$ will have at most two nodes in common. Since $|\Gamma^{-1}A| = |\Gamma^{-1}B| = r \geqslant 3$, we have $\Gamma^{-1}A - B \neq \Gamma^{-1}B - A$. For $B = \{v_i\}$ such that $v_i = 0\ldots0\underbrace{1\ldots1}_{k}$, $k \geqslant 3$, again we have $\Gamma^{-1}A - B = \Gamma^{-1}A$, and $\Gamma^{-1}B - A = \Gamma^{-1}B$. Since $\Gamma^{-1}A \neq \Gamma^{-1}B$, we have $\Gamma^{-1}A - B \neq \Gamma^{-1}B - A$.

This completes the proof of the theorem. $\square$

Corollary 2. *An n-cube system is n-diagnosable,* $n \geqslant 3$, *under the BGM model.*

Using Theorem 3 and the BGM diagnosability algorithm in Ref. [12], the diagnosability algorithm for incomplete hypercubes under BGM model is very similar to that for PMC model, presented in the previous section, with three main steps. The only difference is at Step 3, when the minimum degree of all nodes is less than or equal to 2. In that event, the BGM diagnosability algorithm of Ref. [12] will be called to compute the system's diagnosability. Again, we point out that for a hypercube of reasonably large size, we really should anticipate that the minimum degree is bigger than 2 most of the time, so that the algorithm of Ref. [12] is executed very infrequently.

For the time complexity: The BGM diagnosability algorithm in Ref. [12] has complexity $O(|V|\tau^2/\log\tau)$, where τ is the system's diagnosability. Since $\tau \leqslant 2$ when this algorithm is called, we have its complexity $O(|V|)$. Therefore the total complexity to compute an incomplete hypercube's diagnosability is $O(|V|)$.

5. Conclusion

The problem of determining diagnosability for incomplete hypercubes, i.e., hypercubes that have arbitrarily distributed missing links, is studied in this paper. It is shown that for both the classic PMC diagnostic model and its variant, the BGM model, an incomplete hypercube's diagnosability is equal to the minimum number of functioning links of a node if the minimum number $\geqslant 3$. If the minimum number $\leqslant 2$, the algorithms developed in Ref. [12] are called to decide the diagnosability, which will run in linear time for small minimum degree. The result of Theorem 2 gives a more generalized presentation of the well-known n-diagnosability of n-cubes.

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